GMS 90 Final Exam Solutions

Dec 15, 2011

See solutions to problems 1–20 in Hawkes Learning System.

21. (15 pts) Solve the equation

$$11 - 3x = 3 + x$$

for the real number x. Justify each step by referring to an appropriate algebraic property. Be sure not to skip or combine steps.

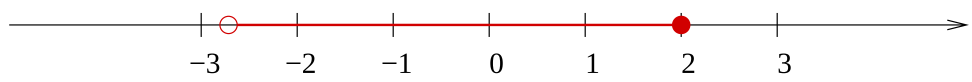
$$\begin{aligned}
11 - 3x &= 3 + x & & \\
11 + (-3x) &= 3 + x & & \text{definition of subtraction} \\
(11 + (-3x)) + 3x &= (3 + x) + 3x & & \text{addition property of equality} \\
11 + (-3x + 3x) &= 3 + (x + 3x) & & \text{associativity of addition} \\
11 + 0 &= 3 + (x + 3x) & & \text{inverse property of addition} \\
11 &= 3 + (x + 3x) & & \text{identity property of addition} \\
11 &= 3 + (1 + 3)x & & \text{distributivity of multiplication over addition} \\
11 &= 3 + 4x & & \text{arithmetic} \\
-3 + 11 &= -3 + (3 + 4x) & & \text{addition property of equality} \\
8 &= -3 + (3 + 4x) & & \text{arithmetic} \\
8 &= (-3 + 3) + 4x & & \text{associativity of addition} \\
8 &= 0 + 4x & & \text{inverse property of addition} \\
8 &= 4x & & \text{identity property of addition} \\
\frac{1}{4}8 &= \frac{1}{4}(4x) & & \text{multiplication property of equality} \\
2 &= \frac{1}{4}(4x) & & \text{arithmetic} \\
2 &= \left(\frac{1}{4}4\right)x & & \text{associativity of multiplication} \\
2 &= 1x & & \text{inverse property of multiplication} \\
2 &= x & & \text{identity property of multiplication}
\end{aligned}$$

22. (a) (10 pts) Find all real solutions x of the compound inequality

$$-4 \leq 2 - 3x < 10.$$

$$\begin{gathered}
-4 \leq 2 - 3x < 10 \\
-4 - 2 \leq 2 - 3x - 2 < 10 - 2 \\
-6 \leq -3x < 8 \\
\frac{-6}{-3} \geq \frac{-3x}{-3} > \frac{8}{-3} \\
2 \geq x > -\frac{8}{3}
\end{gathered}$$

(b) (2 pts) Plot the solution set on the real line.



(c) (3 pts) Describe the solution set in set builder notation.

$$\left\{x \in \mathbb{R} \mid -\frac{8}{3} < x \leq 2\right\}$$

23. (10 pts) Prove the Zero Factor Law for real numbers: $0 \cdot x = 0$ for all real numbers x.

Let $y = 0 \cdot x$. Then

$$y = 0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x = y + y$$

where we used $0 = 0+0$ and the distributive law. Therefore $y = y + y$. Now subtract y from both sides to get $0 = y = 0 \cdot x$.

24. (10 pts) Last year Santa spent 20,000 euros on reindeer feed and 100,000 euros on his elves's salary. Since then, Santa's budget has been cut by 10,400 euros. To deal with the cut, Santa has reduced his expenditure on reindeer feed by switching to a cheaper brand, and has renegotiated his elves' salaries with the North Pole Elf Union. His percent savings on the feed are 4% more than his savings on the elves' salaries. By what percentages did Santa reduce his spending on feed and elf salaries?

Let r be the rate by which Santa reduced the elves' salaries. Then his percent savings on the feed is $r + 4\% = r + 0.04$. So Santa would save $r100,000$ on elf salaries and $(r+0.04)20,000$ on reindeer feed. Hence

$$\begin{aligned}
r100,000 + (r+0.04)20,000 &= 10,400 \\
r100,000 + r20,000 + 0.04(20,000) &= 10,400 \\
120,000r + 800 &= 10,400 \\
120,000r + 800 - 800 &= 10,400 - 800 \\
120,000r &= 9,600 \\
\frac{120,000r}{120,000} &= \frac{9,600}{120,000} \\
r &= 0.08
\end{aligned}$$

So Santa reduced the elves' salaries by 8% and his spending on feed by 12%.

25. (10 pts) **Extra credit problem.** In class, we showed that addition and multiplication of integer numbers are both commutative. Use these facts to show that addition of rational numbers (fractions) is also commutative. (Hint: you can argue using a concrete example of numbers if you wish, but make sure your example is generic enough that your argument applies to any other pair of rational numbers.)

Let $x, y \in \mathbb{Q}$. We need to show $x+y=y+x$. Since $x, y \in \mathbb{Q}$, we know that there exist some integers m, n, p, q such that $n, p \neq 0$ and $x=m/n$ and $y=p/q$. Now

$$\begin{aligned}
x+y &= \frac{m}{n}+\frac{p}{q} \\
&= \frac{m}{n}\frac{q}{q}+\frac{p}{q}\frac{n}{n} \\
&= \frac{mq}{nq}+\frac{pn}{qn} \\
&= \frac{mq}{nq}+\frac{pn}{nq} && \text{since } qn=nq \\
&= \frac{mq+pn}{nq}
\end{aligned}$$

where we used $qn=nq$ by the commutitativity of multiplication on integers. Also

$$\begin{aligned}
y+x &= \frac{p}{q}+\frac{m}{n} \\
&= \frac{p}{q}\frac{n}{n}+\frac{m}{n}\frac{q}{q} \\
&= \frac{pn}{qn}+\frac{mq}{nq} && \text{since } qn=nq \\
&= \frac{pn+mq}{nq}
\end{aligned}$$

Since mq and np are integer numbers and we know addition is commutative on the integers, $mq+np=np+mq$. Therefore

$$x+y=\frac{mq+np}{nq}=\frac{np+mq}{nq}=y+x.$$