GMS 91 Exam 2 Solutions

Nov 8, 2012

See solutions to problems 1–8 in Hawkes Learning System.

9. (10 pts) The volume that a certain amount of gas occupies is directly proportional to its temperature (measured in Kelvin) and inversely proportional to its pressure. A weather balloon filled with 1000 l of helium is released in the Imperial Valley, where the temperature is 300 K (27°C). As the balloon rises, the temperature and pressure decrease. By the time the balloon reaches the height of Mt Everest (29,029 ft), its temperature is 240 K (−33°C). The pressure at that altitude is one third of the pressure in the Imperial Valley. What is the volume of the balloon at this altitude (29029 ft)?

   Let V be the volume, T the temperature, and p the pressure. We are told $V = kT/p$ for some constant k. We don't know the pressure in the Imperial Valley, but we can call it p_0. Then the pressure at 29029 ft is $p_0/3$. We know

$$1000 = \frac{300k}{p_0} \implies k = \frac{1000}{300}p_0 = \frac{10}{3}\,p_0.$$

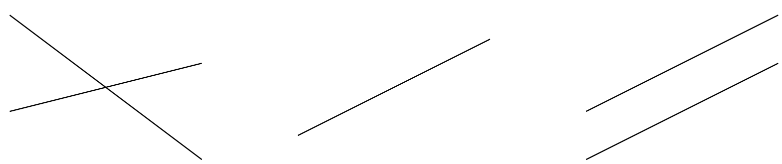
   The same balloon at 29029 ft will have volume

$$V = \frac{240k}{\frac{p_0}{3}} = \frac{240\,\frac{10}{3}\,p_0}{\frac{p_0}{3}} = 2400.$$

   So the balloon has a volume of 2400 l at 29029 ft.

10. (5 pts) Explain why a system of two linear equations in two variables has either exactly one solution, or infinitely many solutions, or no solution at all. (Hint: think lines.)

    The solution set of a linear equation in two variables is a line in the plane. Two such equations would describe two lines in the plane. The solution set of the system is the intersection of those two lines. Two lines in the plane can either intersect at exactly one point, or be the same line, or be parallel:



    If they intersect at one point, that point is the only solution of the system. If they are the same line, their intersection is still that same line, which has infinitely many points on it. If they are parallel, they do not intersect, so the system has no solution.

11. (10 pts) I have a total of \$72 in \$10 bills, \$5 bills, and \$1 bills. If I changed all of my \$10 bills into \$1 bills, I would have twice as many bills as before. If I instead changed all of my \$5 bills into \$1 bills, I would have three times as many bills as what I started with. How many of each kind of bill do I have? Check your answer.

    Let

$$x = \text{number of \$10 bills}$$
$$y = \text{number of \$5 bills}$$
$$z = \text{number of \$1 bills}$$

    Then we have the following

	Initially	If \$10 bills are changed to \$1 bills	If \$5 bills are changed to \$1 bills
number of \$10 bills	x	0	x
number of \$5 bills	y	y	0
number of \$1 bills	z	$z+10x$	$z+5y$
total number of bills	$x+y+z$	$10x+y+z$	$x+5y+z$
total amount of money	$10x+5y+z$	$10x+5y+z$	$10x+5y+z$

So we get the following equations:

$$10x+5y+z=72 \tag{1}$$

$$10x+y+z=2(x+y+z) \tag{2}$$

$$x+5y+z=3(x+y+z) \tag{3}$$

From Eq (3),

$$\begin{aligned} x+5y+z &= 3x+3y+3z \\ 0 &= 2x-2y+2z \\ 0 &= x-y+z \\ y &= x+z \end{aligned}$$

From Eq (2),

$$\begin{aligned} 10x+y+z &= 2x+2y+2z \\ 8x-y-z &= 0 && \text{substitute } y=x+z \\ 8x-(x+z)-z &= 0 \\ 7x-2z &= 0 \\ 7x &= 2z \\ \frac{7}{2}x &= z \end{aligned}$$

So $y=x+z=x+(7/2)x=(9/2)x$. Now we use Eq (1),

$$\begin{aligned} 10x+5y+z &= 72 && \text{subtitute } y=\frac{9}{2}x \text{ and } z=\frac{7}{2}x \\ 10x+5\left(\frac{9}{2}x\right)+\frac{7}{2}x &= 72 \\ \frac{20+45+7}{2}x &= 72 \\ 36x &= 72 \\ x &= 2 \end{aligned}$$

So $y=(9/2)2=9$ and $z=(7/2)2=7$.

Hence I had two \$10 bills, nine \$5 bills, and seven \$1 bills.

Check: Two \$10 bills, nine \$5 bills, and seven \$1 bills are worth a total of $20+45+7=72$ dollars. $\surd$

I have $2+9+7=18$ bills. If I changed the two \$10 bills I'd get 20 \$1 bills. So I'd have $20+9+7=36$ bills. That's indeed twice as many as I used to have. $\surd$

If I changed the nine \$5 bills I'd get 45 \$1 bills. So I'd have $2+45+7=54$ bills. That's indeed three times as many as I used to have. $\surd$

12. (10 pts) **Extra credit problem.** Can you find a system of two linear inequalities in x and y which has exactly one ordered pair as its solution? If so, find one; if not, explain why no such system can exist.

There is no such system. The solution set of a linear equation in two variables is a half-plane, which may or may not include its boundary line. The solution set of two such equations is the intersection of two half-planes. But there is no way for two half-planes to intersect in exactly one point. Either they do not intersect at all, or their intersection is wedge, or a half plane, or a band between two parallel lines, all of which contain infinitely many points.