GMS 91 Final Exam Solutions

Dec 13, 2012

See solutions to problems 1–16 in Hawkes Learning System.

17. (10 pts) Let x, y, z be real numbers. Explain why a system of linear equations in x, y, z must have either one solution, or infinitely many solutions, or no solution at all.

    The solution set of each equation is a plane in 3-dimensional space. The solution set of the system of equations is the intersection of all these planes. Notice that any two planes either intersect in a line, or if they are parallel, they do not intersect at all. Now, there are four possibilities. You can see some of these in Figure 3 on p. 636 in your textbook.

    17. It could be that the planes do not have a common intersection, for example if two of them are parallel. In this case, the system of equations has no solution.
    17. The planes could all go through a common point, e.g. if two of them intersect in a line and the third one cuts through this line. In this case, the system has exactly one solution.
    17. The planes could all go through a common line, e.g. if they look like the paddles of a paddle wheel. In this case, every point on that line is a solution to the system of equations, and there are infinitely many of them.
    17. The planes could all be the same. In this case, their intersection is a plane, which of course has infinitely many points, so the system has infinitely many solutions.

18. (10 pts) In order to complete all of their preparations by Christmas, Santa Claus and his elves are working around the clock. Santa works morning and afternoon shifts, and the elves work afternoon and night shifts. When they are all working at the same time, it takes them 3 minutes to fill a bag with toys. Santa, working by himself, takes 8 minutes more to fill a bag than the elves do working without Santa. How long does Santa working alone take to fill a bag with toys?

Santa at the end of a double shift

Let x be the time it takes the elves fill a bag with toys. Then we have the following:

	Time to fill a bag	Part of the bag filled in 1 min
Elves by themselves	x	$\frac{1}{x}$
Santa by himself	$x+8$	$\frac{1}{x+8}$
Santa with the elves	3	$\frac{1}{x}+\frac{1}{x+8}$

This gives us the equation

$$\begin{aligned}
\frac{1}{x}+\frac{1}{x+8} &= \frac{1}{3} && \text{Assume } x \neq 0 \text{ and } x+8 \neq 0 \text{, and multiply by } 3x(x+8). \\
3(x+8)+3x &= x(x+8) \\
3x+24+3x &= x^2+8x \\
x^2+8x-3x-3x-24 &= 0 \\
x^2+2x-24 &= 0 \\
(x+6)(x-4) &= 0
\end{aligned}$$

So $x=-6$ or $x=4$. But $x=-6$ does not make sense in the context of this problem. When we multiplied by x and $x+8$, we assumed they were not 0. In fact, if $x=4$, then $x\neq 0$ and $x+8=12\neq 0$. So the assumption was valid. If the elves by themselves take 4 min to fill a bag, then Santa takes 12 min to fill a bag.

I will check my solution just in case. In 3 min, the elves fill 3/4 of a bag, and Santa fills 3/12 of a bag. And $3/4+3/12=3/4+1/4=1$ bag, as expected.

19. (10 pts) Find all real solutions of the equation

$$\sqrt{2x+1}-\sqrt{1+x}=2$$

Be sure to check your solutions.

$$\begin{aligned}
\sqrt{2x+1}-\sqrt{1+x}&=2 && \text{Add } \sqrt{1+x} \text{ to both sides.}\\
\sqrt{2x+1}&=2+\sqrt{1+x} && \text{Square both sides.}\\
(\sqrt{2x+1})^2&=(2+\sqrt{1+x})^2\\
2x+1&=4+4\sqrt{1+x}+(1+x)\\
2x+1-4-(1+x)&=4\sqrt{1+x}\\
x-4&=4\sqrt{1+x} && \text{Square both sides.}\\
(x-4)^2&=(4\sqrt{1+x})^2\\
x^2-8x+16&=16(1+x)\\
x^2-8x+16&=16+16x\\
x^2-24x&=0\\
x(x-24)&=0
\end{aligned}$$

So $x=0$ or $x=24$. Since we squared both sides of the equation twice, we may have introduced false roots. So we need to check if these are really solutions.

$$\sqrt{2\cdot 0+1}-\sqrt{1+0}=\sqrt{1}-\sqrt{1}=0 \qquad \times$$
$$\sqrt{2\cdot 24+1}-\sqrt{1+24}=\sqrt{49}-\sqrt{25}=7-5=2 \qquad \surd$$

So $x=0$ was a false root, and $x=24$ is the only real solution of this equation.

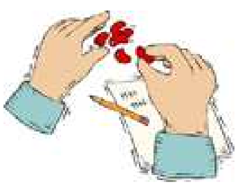
20. (10 pts) Rationalize the denominator of

$$\frac{3}{\sqrt{10}-\sqrt{5}+\sqrt{2}}$$

$$\begin{aligned}
\frac{3}{\sqrt{10}-\sqrt{5}+\sqrt{2}}&=\frac{3}{\sqrt{10}-\sqrt{5}+\sqrt{2}}\cdot\frac{\sqrt{10}-\sqrt{5}-\sqrt{2}}{\sqrt{10}-\sqrt{5}-\sqrt{2}}\\
&=\frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})}{(\sqrt{10}-\sqrt{5})^2-\sqrt{2}^2}\\
&=\frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})}{10+2\sqrt{50}+5-2}\\
&=\frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})}{13+10\sqrt{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})}{13+10\sqrt{2}} \cdot \frac{13-10\sqrt{2}}{13-10\sqrt{2}} \\
&= \frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})(13-10\sqrt{2})}{13^2-(10\sqrt{2})^2} \\
&= \frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})(13-10\sqrt{2})}{169-100\cdot 2} \\
&= -\frac{3(\sqrt{10}-\sqrt{5}-\sqrt{2})(13-10\sqrt{2})}{31}
\end{aligned}
$$

21. (10 pts) **Extra credit problem.** Recall the game of Fibonacci Nim, which goes as follows. Two players, let's call them Alice and Bob, play against each other by taking turns removing objects, say beans, from a pile. Alice goes first and she may remove as many beans as she wants, except for all of them. Bob can then remove up to twice the number of beans Alice has just removed. Now it is Alice's turn, and she can remove up to twice the number of beans Bob has just removed. And so on, each time a player can remove up to twice the number of beans the other player removed in the preceding turn. The player that removes the last bean wins. For example, if the pile initially has 5 beans, and Alice removes 1 on her first move, then Bob can take 1 or 2. If Bob takes 2, Alice can now take 1–4, except of course that there are only 2 beans left. By taking those 2, Alice wins the game.



Show that if the pile initially has 9 beans, Alice can win no matter what Bob does. (Hint: Try starting with 2 beans, 3 beans, 4 beans, etc to understand who has a winning strategy in each case.)

Starting with 2 beans, Alice is bound to lose, because her only legal move is to take one of them, and then Bob will take the other.

Starting with 3 beans, Alice will again lose, because whether she starts by taking one or two beans, Bob can take the rest.

Starting with 4 beans, Alice has a winning strategy. If she takes one bean, than Bob will face a pile of 3, and he can only take one or two. Whether he takes one or two, Alice can take the rest. In fact, notice that the reason Bob loses this game is the same why Alice would lose if the pile initially had only 3 beans.

Starting with 5 beans, Alice will lose. If she takes two, three, or four, Bob will take the rest. If she takes one, then Bob has a pile of 4, and he can win using the same strategy Alice would use to win if starting with 4 beans.

Starting with 6 beans, Alice has a winning strategy. She can take just one bean, and put Bob in the same kind of losing situation she was in when starting with 5 beans.

Starting with 7 beans, Alice again has a winning strategy. She can take two beans, and put Bob in the same kind of losing situation she was in when starting with 5 beans.

Starting with 8 beans, Alice will lose. If she takes three or more beans, Bob can take the rest. If she takes one or two, she leaves Bob with the same kind of winning situation she was in when they started with 6 or 7 beans.

Starting with 9 beans, Alice has a winning strategy. If she takes one bean, she will put Bob in the same kind of losing situation she was in when they started with 8 beans.