GMS 91 Exam 1 Solutions

Oct 13, 2008

1. (4 pts each) Do the following computations with complex numbers. You don't need to explain why you do the computations the way you do, but be sure to show the details of your work.

   (a)

$$(2-5i)(-1+3i) = 2(-1)+2(3i)-(5i)(-1)-(5i)(3i) =$$
$$= -2+6i+5i-15i^2 = -2+11i+15 = 13+11i$$

   (b)

$$\frac{2-5i}{-1+3i} = \frac{2-5i}{-1+3i}\frac{\overline{-1+3i}}{\overline{-1+3i}} = \frac{(2-5i)(-1-3i)}{(-1+3i)(-1-3i)}$$
$$= \frac{2(-1)+2(-3i)-(5i)(-1)-(5i)(-3i)}{(-1)^2+3^2}$$
$$= \frac{-2-6i+5i-15}{10} = \frac{-17-i}{10} = -\frac{17}{10}-\frac{i}{10}$$

   (c)

$$|7-4i| = \sqrt{(7-4i)(7+4i)} = \sqrt{7^2+(-4)^2} = \sqrt{49+16} = \sqrt{65}$$

   (d) i^{-15}

   Since $i^4 = i^2 i^2 = (-1)(-1) = 1$,

$$i^{-15} = \frac{1}{i^{15}} = \frac{i}{i^{16}} = \frac{i}{1^4} = i.$$

2. (5 pts each) Find the following greatest common divisors. You don't need to explain why you do the computations the way you do, but be sure to show the details of your work.

   (a)

$$\gcd(108,144) = \gcd(2^2\cdot 3^3, 2^4\cdot 3^2) = 2^2\cdot 3^2 = 36$$

   (b)

$$\gcd(3^5\cdot 5^4\cdot 13^2, 2^3\cdot 3^2\cdot 5\cdot 13^4, 3\cdot 5^2\cdot 13) = 3\cdot 5\cdot 13 = 195$$

   (c) $\gcd(3^4\cdot 7^2\cdot 11, 2^3\cdot 5^6) = 1$ since the two numbers have no prime factors in common.

3. (5 pts each) Let $z = x+yi$ be any complex number.

   (a) Find $|z|$.

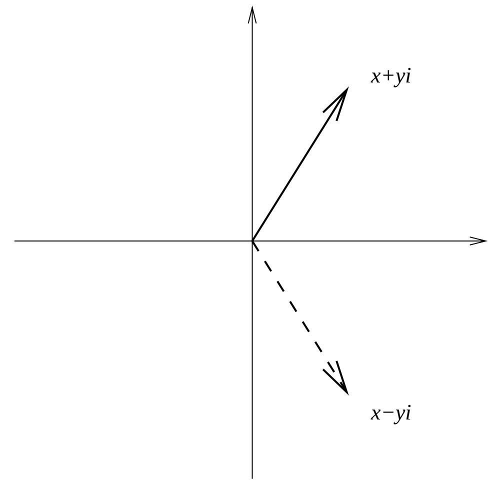
$$|z| = \sqrt{z\overline{z}} = \sqrt{(x+yi)(x-yi)} = \sqrt{x^2+y^2}$$

   (b) Find $|\overline{z}|$.

$$|\overline{z}| = |\overline{x+yi}| = |x-yi| = \sqrt{(x-yi)(x+yi)} = \sqrt{x^2+y^2}$$

   (c) Compare the results of (a) and (b). Explain why this happens.

They are the same. The absolute value of a complex number is its distance from 0 in the complex plane. A number and its conjugate are mirror images of each other across the real axis. So their distances from 0 must be the same.



4. (a) (4 pts) Explain what it means that multiplication on the integers is associative. Give an example of this. (You are not asked to prove that multiplication is associative, only to explain what the term means.)

   It means that $(xy)z = x(yz)$ for any three integers x, y, z. For example,

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$$
$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24.$$

   (b) (4 pts) Explain what it means that multiplication on the rational numbers is associative. Give an example of this. (You are not asked to prove that multiplication is associative, only to explain what the term means.)

   It means that $(xy)z = x(yz)$ for any three rational numbers x, y, z. For example,

$$\left(\frac{2}{3}\frac{5}{7}\right)\frac{9}{4} = \frac{10}{21}\frac{9}{4} = \frac{90}{84}$$
$$\frac{2}{3}\left(\frac{5}{7}\frac{9}{4}\right) = \frac{2}{3}\frac{45}{28} = \frac{90}{84}.$$

   (c) (6 pts) Assume that multiplication is associative on the integers. Use this to justify that multiplication is also associative on the rational numbers. (You are now asked to prove that multiplication is associative on the rationals.)

   Let $x, y, z \in \mathbb{Q}$. Then there exist integers m, n, p, q, r, s such that $n, q, s \neq 0$ and $x = m/n$, $y = p/q$, and $z = r/s$.

$$(xy)z = \left(\frac{m}{n}\frac{p}{q}\right)\frac{r}{s} = \frac{mp}{nq}\frac{r}{s} = \frac{(mp)r}{(nq)r}$$
$$x(yz) = \frac{m}{n}\left(\frac{p}{q}\frac{r}{s}\right) = \frac{m}{n}\frac{pr}{qs} = \frac{m(pr)}{n(qr)}$$

   Since multiplication is associative on the integers $(mp)r = m(pr)$ and $(nq)s = n(qs)$. Therefore

$$(xy)z = \frac{(mp)r}{(nq)r} = \frac{m(pr)}{n(qr)} = x(yz).$$

5. **Extra credit problem.** The quadratic equation $x^2 - 3x + 5 = 0$ has no real solutions. But it has two complex solutions. You can find them using the usual quadratic formula.

(a) (4 pts) Find the two solutions of $x^2 - 3x + 5 = 0$.

The solutions are

$$\begin{aligned} \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)}}{2} &= \frac{3 \pm \sqrt{9-20}}{2} = \frac{3 \pm \sqrt{-11}}{2} \\ &= \frac{3 \pm \sqrt{11}\sqrt{-1}}{2} = \frac{3 \pm \sqrt{11}i}{2} = \frac{3}{2} \pm \frac{\sqrt{11}}{2}i \end{aligned}$$

(b) (4 pts) Let A and B denote the two solutions you found in part (a). Verify (by doing the actual computation) that $(x-A)(x-B) = x^2 - 3x + 5$.

$$\begin{aligned} \left(x - \frac{3+\sqrt{11}i}{2}\right)\left(x - \frac{3-\sqrt{11}i}{2}\right) &= x^2 - \frac{3+\sqrt{11}i}{2}x - \frac{3-\sqrt{11}i}{2}x + \frac{3+\sqrt{11}i}{2}\frac{3-\sqrt{11}i}{2} \\ &= x^2 - \frac{3+\sqrt{11}i+3-\sqrt{11}i}{2}x + \frac{3^2+\sqrt{11}^2}{4} \\ &= x^2 - \frac{6}{2}x + \frac{9+11}{4} = x^2 - 3x + 5 \end{aligned}$$

(c) (7 pts) Compute $A+B$ and AB. What do these have to do with the original equation? Explain why this happens.

Actually, notice we already computed $A+B$ and AB in part (b):

$$\begin{aligned} A+B &= \frac{3+\sqrt{11}i}{2} + \frac{3-\sqrt{11}i}{2} = 3 \\ AB &= \frac{3+\sqrt{11}i}{2}\frac{3-\sqrt{11}i}{2} = 5 \end{aligned}$$

Notice that $-(A+B) = -3$ is the coefficient of x and $AB = 5$ is the constant term in the original equation. This must be so since

$$(x-A)(x-B) = x^2 - Ax - Bx + AB = x^2 - (A+B)x + AB.$$

Incidentally, the same is true when the roots are real and that is standard material in Algebra I together with the quadratic formula.