GMS 91 Final Exam Solutions

Dec 17, 2008

1. (3 pts each) Do the following computations with complex numbers. You don't need to explain why you do the computations the way you do, but be sure to show the details of your work.

   (a) $\overline{4-9i}$

$$\overline{4-9i} = 4+9i$$

   (b) $|4-9i|$

$$|4-9i| = \sqrt{(4-9i)(4+9i)} = \sqrt{4^2-9^2i^2} = \sqrt{16+81} = \sqrt{97}$$

   (c) $\dfrac{6+5i}{4-9i}$

$$\frac{6+5i}{4-9i} = \frac{6+5i}{4-9i}\frac{4+9i}{4+9i} = \frac{24+54i+20i+45i^2}{4^2-9^2i^2} = \frac{-21+74i}{97}$$

   (d) i^{-42}

$$i^{-42} = \frac{1}{i^{42}} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

2. (10 pts) Solve the equation

$$|4x-8| = |7x-5|.$$

Since $|4x-8| = \pm(4x-8)$ and $|7x-5| = \pm(7x-5)$, we need to consider 4 cases depending on the signs of $4x-8$ and $7x-5$.

Case $4x-8 \geq 0$ and $7x-5 \geq 0$: In this case $|4x-8| = 4x-8$ and $|7x-5| = 7x-5$. So we get the equation

$$\begin{aligned} 4x-8 &= 7x-5 \\ -3 &= 3x \\ x &= -1 \end{aligned}$$

We need to make sure that the solution $x=-1$ satisfies the conditions for this case. Actually $4x-8 = 4(-1)-8 = -12 < 0$, so it does not. Therefore we reject this solution.

Case $4x-8 \geq 0$ and $7x-5 < 0$: In this case $|4x-8| = 4x-8$ and $|7x-5| = -(7x-5) = 5-7x$. So we get the equation

$$\begin{aligned} 4x-8 &= 5-7x \\ 11x &= 13 \\ x &= \frac{13}{11} \end{aligned}$$

We need to make sure that the solution $x = 13/11$ satisfies the conditions for this case. Actually $4(13/11)-8 < 52/11-88/11 < 0$, so it does not. Therefore we reject this solution.

Case $4x - 8 < 0$ **and** $7x - 5 \geq 0$**:** In this case $|4x-8| = -(4x-8) = 8-4x$ and $|7x-5| = 7x-5$. So we get the equation

$$8 - 4x = 7x - 5$$

which is equivalent to the equation in the previous case, hence its solution is $x = 13/11$. We already know $4(13/11) - 8 < 0$. Also $7(13/11) - 5 > 7 - 5 > 0$, so $13/11$ does indeed fit this case.

Case $4x - 8 < 0$ **and** $7x - 5 < 0$**:** In this case $|4x-8| = -(4x-8) = 8-4x$ and $|7x-5| = -(7x-5) = 5-7x$. So we get the equation

$$8 - 4x = 5 - 7x$$

which is equivalent to the equation in the first case, hence its solution is $x = -1$. We already know $4(-1) - 8 < 0$. Also $7(-1) - 5 = -12 < 0$, so -1 does indeed fit this case.

We can conclude that the solutions of the original equation are -1 and $13/11$.

I will check my answer for good measure:

$$|4(-1) - 8| = |-12| = 12 \text{ and } |7(-1) - 5| = |-12| = 12 \qquad \surd$$

$$\left|4\frac{13}{11} - 8\right| = \left|\frac{52-88}{11}\right| = \left|-\frac{36}{11}\right| = \frac{36}{11} \text{ and } \left|7\frac{13}{11} - 5\right| = \left|\frac{91-55}{11}\right| = \left|\frac{36}{11}\right| = \frac{36}{11} \qquad \surd$$

3. (a) (8 pts) Find the quotient and remainder using long division for

$$\frac{2x^3 - 14x^2 + 7x - 31}{2x^2 + 5}.$$

$$\begin{array}{r|l}
 & \quad\;\; x \quad -7 \\
\hline
2x^2+5 & 2x^3 - 14x^2 + 7x - 31 \\
 & -2x^3 \qquad\quad\;\; -5x \\
\hline
 & \quad -14x^2 + 2x - 31 \\
 & \quad\;\; 14x^2 \qquad\; +35 \\
\hline
 & \qquad\qquad\; 2x \;\; +4
\end{array}$$

So the quotient is $x - 7$ and the remainder is $2x + 4$.

(b) (4 pts) Check you answer to part (a) by doing a multiplication and an addition of polynomials. Does it work?

$$\begin{aligned}
(x-7)(2x^2+5) + (2x+4) &= 2x^3 + 5x - 14x^2 - 35 + 2x + 4 \\
&= 2x^3 - 14x^2 + 7x - 31 \qquad \surd
\end{aligned}$$

4. (10 pts) Simplify the expression

$$\frac{2x^2 + 5x + 2}{x^2 + 2x - 3} \Big/ \frac{x^2 + 3x + 2}{2x^2 - 3x + 1}.$$

(If you have time, try to think of a way to check your answer.)

First,

$$\frac{2x^2 + 5x + 2}{x^2 + 2x - 3} \Big/ \frac{x^2 + 3x + 2}{2x^2 - 3x + 1} = \frac{2x^2 + 5x + 2}{x^2 + 2x - 3} \, \frac{2x^2 - 3x + 1}{x^2 + 3x + 2}.$$

Now, it's helpful to factor all of the quadratic polynomials. One way to do this is by intelligent guessing, another is by finding the roots using the quadratic formula. Here is an example of each. If

$$x^2 + 2x - 3 = (x + a)(x + b)$$

then $a + b = 2$ and $ab = -3$. It is easy to spot the solution as $a = -1$, $b = 3$, or the other way around. (If you don't see this, you can always solve this system of equations by substitution, but you will have to solve the same quadratic equation on the way as the original polynomial.) So

$$x^2 + 2x - 3 = (x - 1)(x + 3).$$

Now

$$2x^2 + 5x + 2 = 2\left(x^2 + \frac{5}{2}x + 1\right)$$

If you don't see how to factor this, the quadratic formula gives the roots

$$\begin{aligned} x_{1,2} &= \frac{-\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 4}}{2} = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{16}{4}}}{2} = \frac{-\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2} \\ &= \frac{-\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2} = -\frac{1}{2}, -2. \end{aligned}$$

So

$$2x^2 + 5x + 2 = 2\left(x + \frac{1}{2}\right)(x + 2) = (2x + 1)(x + 2).$$

You can get

$$\begin{aligned} 2x^2 - 3x + 1 &= (2x - 1)(x - 1) \\ x^2 + 3x + 2 &= (x + 1)(x + 2) \end{aligned}$$

similarly. So

$$\begin{aligned} \frac{2x^2 + 5x + 2}{x^2 + 2x - 3}\,\frac{2x^2 - 3x + 1}{x^2 + 3x + 2} &= \frac{(2x + 1)(x + 2)}{(x - 1)(x + 3)}\,\frac{(2x - 1)(x - 1)}{(x + 1)(x + 2)} \\ &= \frac{(2x + 1)(2x - 1)}{(x + 1)(x + 3)} \end{aligned}$$

One way to sort of check this answer is to substitute a few numbers into the original expression and this result and see if the evaluate to the same thing. This is not foolproof, two different expressions could give the same value for the same value of x, but if you try a few numbers and the results always match, you can have some confidence in your computation. The more numbers give matching results, the more confident you can be. So here are a few easy numbers:

$x = 0$:

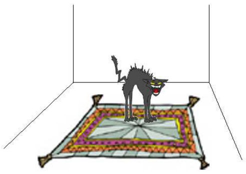
$$\begin{aligned} \frac{2x^2 + 5x + 2}{x^2 + 2x - 3} \Big/ \frac{x^2 + 3x + 2}{2x^2 - 3x + 1} &= \frac{2}{-3} \Big/ \frac{2}{1} = -\frac{1}{3} \\ \frac{(2x + 1)(2x - 1)}{(x + 1)(x + 3)} &= \frac{1(-1)}{1 \cdot 3} = -\frac{1}{3} \end{aligned}$$

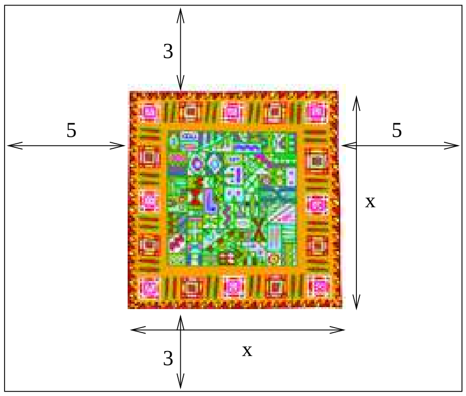
$x = 2$:

$$\begin{aligned} \frac{2x^2 + 5x + 2}{x^2 + 2x - 3} \Big/ \frac{x^2 + 3x + 2}{2x^2 - 3x + 1} &= \frac{8 + 10 + 2}{4 + 4 - 3} \Big/ \frac{4 + 6 + 2}{8 - 6 + 1} = \frac{20}{5} \Big/ \frac{12}{3} = \frac{4}{4} = 1 \\ \frac{(2x + 1)(2x - 1)}{(x + 1)(x + 3)} &= \frac{5 \cdot 3}{3 \cdot 5} = 1 \end{aligned}$$

Indeed, they both match.

5. (6 pts) A square rug lies in the middle of a rectangular room. There are 3 feet of uncovered floor on two opposite sides of the rug and 5 feet of uncovered floor on the other two opposite sides. Find a polynomial expression for the area of the room in terms of x, the side length of the rug.





To the left is a picture of the room from above.
So one side of the room is $x+3+3 = x+6$ and the other is $x+5+5 = x+10$. Therefore the area of the room is $(x+6)(x+10) = x^2+16x+60$.

6. (10 pts) Find the greatest common divisor and the least common multiple of $3^3 \cdot 5 \cdot 11^2$, $2^5 \cdot 3^2 \cdot 7 \cdot 11$, and $2^2 \cdot 5 \cdot 7^2$.

To find the gcd pick the smallest power of each prime from the three numbers. It may help to show the 0th powers too. I underlined the lowest powers below:

$$\gcd(\underline{2^0} \cdot 3^3 \cdot 5 \cdot \underline{7^0} \cdot 11^2,\ 2^5 \cdot 3^2 \cdot \underline{5^0} \cdot 7 \cdot 11,\ 2^2 \cdot \underline{3^0} \cdot 5 \cdot 7^2 \cdot \underline{11^0}) = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 \cdot 11^0 = 1.$$

For the lcm, pick the largest power of each prime. I underlined the highest powers below:

$$\mathrm{lcm}(\underline{3^3} \cdot \underline{5} \cdot \underline{11^2},\ \underline{2^5} \cdot 3^2 \cdot 7 \cdot 11,\ 2^2 \cdot 5 \cdot \underline{7^2}) = 2^5 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11^2.$$

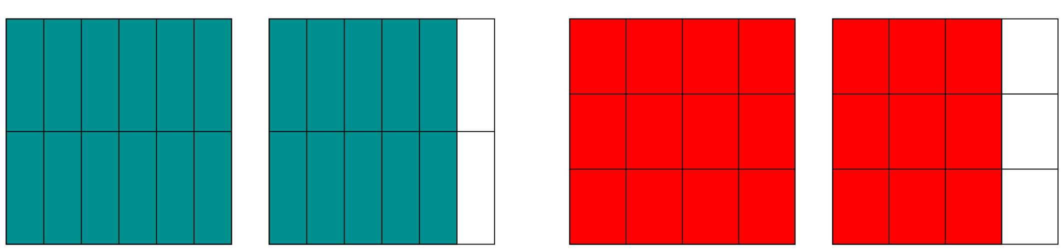
7. (a) (8 pts) Compute

$$\frac{11}{6} + \frac{7}{4}$$

and explain in detail why you can add fractions this way. (Hint: a diagram may help with your explanation.)

$$\frac{11}{6} + \frac{7}{4} = \frac{11}{6}\frac{2}{2} + \frac{7}{4}\frac{3}{3} = \frac{22}{12} + \frac{21}{12} = \frac{43}{12}.$$

This is easy, but how do you explain it? Below are some squares where the shaded parts represent 11/6 and 7/4. Each is further divided along the other side so that there are 12 little rectangles in each square. Even though their shapes are different, each little rectangle is 1/12 of the whole square and there are 43 of them.



$\frac{11}{6}$ $\frac{7}{4}$

(b) (4 pts) What is a rational number?

A rational number is a quotient of an integer by a nonzero integer.

(c) (8 pts) Show that the set of rational numbers is closed under addition, that is when you add two rational numbers, the result is again a rational number.

This argument was on your homework on rational numbers. You can read it there.

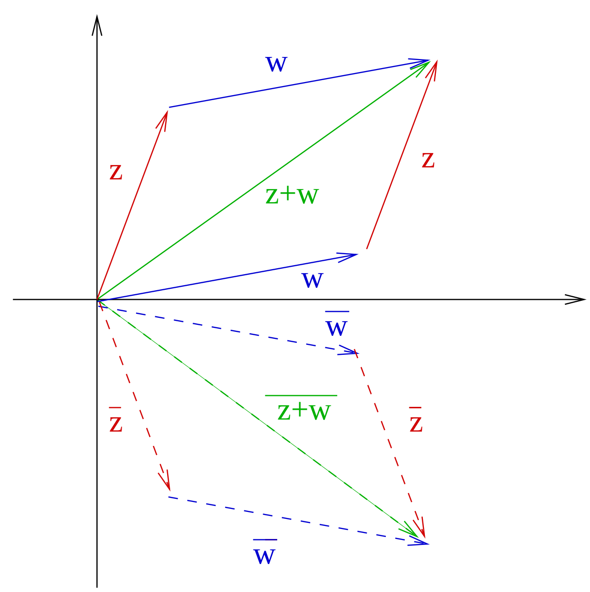
8. (5 pts each) Let $z = x + yi$ and $w = s + ti$ be complex numbers.

(a) Find an algebraic argument to explain why $\overline{z} + \overline{w} = \overline{z + w}$.

$$\begin{aligned}\overline{z} + \overline{w} &= \overline{(x + yi)} + \overline{(s + ti)} = (x - yi) + (s - ti) \\ &= x + s - (y + t)i = \overline{(x + s + (y + t)i)} = \overline{(x + yi + s + ti)} = \overline{z + w}\end{aligned}$$

(b) Now find a geometric argument to explain the same. It may be easier to follow your argument if you illustrate it with a picture.

Look at the picture on the right. We saw in class that adding $z + w$ means adding the arrows of z and w just like in the picture. Now $\overline{z + w}$ is the reflection of this result across the horizontal axis.
Also, $\overline{z}$ and $\overline{w}$ are mirror images of z and w across the horizontal axis. When you add their arrows, the whole addition is the mirror image of $z + w$. So the result is the mirror image of $z + w$ across the horizontal axis, that is $\overline{z + w}$.



9. (a) (4 pts) What is a function? Give an exact definition.

A function f is a rule from a nonempty set A to a set B, which send each element of A to a unique element of B.

(b) (6 pts) Give an example of a function from some set A to some set B, where the elements of A and B not numbers. Be sure to specify what A and B and justify why your example is indeed a function.

Let A be the set kids in the world who have been bad this year and B the set of lumps of coal in Santa's bag. Let f assign to each naughty kid the lump of coal that kid will receive from Santa this year. Then f is a function because to each kid it assigns one and only one lump of coal in Santa's bag.

10. **Extra credit problem.**

(a) (5 pts) Let $a = 300$ and $b = 315$. Find $\gcd(a, b) \operatorname{lcm}(a, b)$. Compare it with ab. What do you notice?

First $300 = 2^2 \cdot 3 \cdot 5^2$ and $315 = 3^2 \cdot 5 \cdot 7$. So $\gcd(a, b) = 3 \cdot 5 = 15$ and $\operatorname{lcm}(a, b) = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 = 6300$. Now

$$15 \cdot 6300 = 94500 = 300 \cdot 315.$$

So $\gcd(a, b) \operatorname{lcm}(a, b)$ and ab are the same!

(b) (10 pts) What you observed in part (a) is in general true for the gcd and the lcm of two positive integers. Looking at the prime factorizations of the numbers in part (a) may give you an idea why. Find an explanation why this relationship holds for any positive integers a and b.

When you use prime factorization to find the gcd of two numbers, you choose the lower power of each prime that shows up in the factorizations and then multiply these together. For the lcm, you choose the higher power of each prime that shows up in the factorizations and then multiply these together. When you multiply the gcd and the lcm together, both the lower and the higher power of each prime factor appears exactly once. When you multiply the two original numbers together, the same thing happens: for each prime factor, one of the numbers will have the lower power, the other will have the higher, so both the lower and the higher powers will be in the product. Since both products contain the same prime powers, they must be equal. Here is how this works in the example of part (a):

$$\begin{aligned} \gcd(a, b) \operatorname{lcm}(a, b) &= (3 \cdot 5)(2^2 \cdot 3^2 \cdot 5^2 \cdot 7) = 3 \cdot 5 \cdot 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \\ ab &= (2^2 \cdot 3 \cdot 5^2)(3^2 \cdot 5 \cdot 7) = 2^2 \cdot 3 \cdot 5^2 \cdot 3^2 \cdot 5 \cdot 7 \end{aligned}$$

where I didn't write the 0th powers of the primes so the products are a little shorter, and 0th powers are 1 anyway.

(c) (10 pts) Do you think that the same relationship holds for the gcd and lcm of three positive integers too? If so, explain why it should hold; if not, find an example which shows that it does not.

Actually, no. Each prime factor will have some power in the prime factorization of each of the three numbers. Then you pick the lowest for the gcd and the highest for the lcm, but the middle one doesn't get picked. The middle power of the prime will still appear in the product of the three numbers, but not in the product of their gcd and lcm. The numbers in problem 6 are a good example. If $a = 3^3 \cdot 5 \cdot 11^2$, $b = 2^5 \cdot 3^2 \cdot 7 \cdot 11$, and $c = 2^2 \cdot 5 \cdot 7^2$, then

$$\gcd(a, b, c) \operatorname{lcm}(a, b, c) = 2^5 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11^2$$

but

$$abc = (3^3 \cdot 5 \cdot 11^2)(2^5 \cdot \underline{3^2} \cdot \underline{7} \cdot \underline{11})(\underline{2^2} \cdot \underline{5} \cdot 7^2) = 2^7 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 11^3.$$

where I underlined those prime powers in the product abc which are missing from the product $\gcd(a, b, c) \operatorname{lcm}(a, b, c)$. You can see why these products are not equal.