GMS 91 Homework solutions on rational numbers

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Since this is homework, you should complete this worksheet individually. As usual with homework, I don't mind if you share ideas with each other, but you copying each other's work is cheating. So don't!

We have talked about the rational numbers in class and have reviewed rational arithmetic in class along with the reasons for it. In this worksheet, you will investigate some properties of the rational numbers.

First, recall the

Definition. A *rational number* is a number of the form m/n where m and n are integers and $n \neq 0$.

As in class, we will denote the set of rational numbers with $\mathbb{Q}$. First, we will look at fundamental properties of addition on $\mathbb{Q}$.

$\mathbb{Q}$ is closed under addition. This means that when we add two rational numbers, the result is also a rational number. Here is the justification. Let x and y be rational numbers. (The shorthand notation for this is to say $x, y \in \mathbb{Q}$. The symbol $\in$ means "element of.") This means that there exist some integers m, n, p, q such that $n, p \neq 0$ and $x = m/n$ and $y = p/q$. We want to convince ourselves that $x + y$ is also a rational number. To be a rational, we must be able to write it as a quotient of two integers (with the denominator not 0 of course). So let's see if that is really possible. We know that to add fractions, we first need to bring them to a common denominator. We can use nq, as it is a common multiple of n and q. So

$$\begin{aligned} x + y &= \frac{m}{n} + \frac{p}{q} \\ &= \frac{m}{n}\frac{q}{q} + \frac{p}{q}\frac{n}{n} \\ &= \frac{mq}{nq} + \frac{pn}{qn} \\ &= \frac{mq + np}{nq} \end{aligned}$$

That certainly looks like a quotient, but is it a quotient of two integers? We know m and q are integers, so their product must also be integer. Similarly, n and p are integers, so their product must also be integer. Hence $mq + np$ is the sum of two integers, and is therefore itself an integer. That takes care of the numerator. The denominator is nq which is a product of two integers and is therefore integer. That's good. But could it be 0? A product is 0 when one of its factors is 0. We know that $n, q \neq 0$, and so their product cannot be 0. This shows $x + y$ is a quotient of two integers, and the one at the bottom is not 0. So $x + y$ is indeed a rational number.

Addition of rational numbers is commutative. That is if $x, y \in \mathbb{Q}$, then $x + y = y + x$. Always. Here is why. Again, since $x, y \in \mathbb{Q}$, we know that there exist some integers m, n, p, q such that $n, p \neq 0$ and $x = m/n$ and $y = p/q$. Now

$$x + y = \frac{mq + np}{nq}$$

as we computed above. What about $y + x$?

$$\begin{aligned} y + x &= \frac{p}{q} + \frac{m}{n} \\ &= \frac{p}{q}\frac{n}{n} + \frac{m}{n}\frac{q}{q} \\ &= \frac{pn}{qn} + \frac{mq}{nq} \\ &= \frac{np + mq}{nq} \end{aligned}$$

Since mq and np are integer numbers and we know addition is commutative on the integers, $mq + np = np + mq$. Therefore

$$x + y = \frac{mq + np}{nq} = \frac{np + mq}{nq} = y + x.$$

Addition of rational numbers is associative. This means that for any $x, y, z \in \mathbb{Q}$, we have $(x + y) + z = x + (y + z)$. You probably don't think very much of this, except possibly "Of course, how could it be any other way?" Well, it could be. There are algebraic systems and operations which are not associative. They are hard to work with. Associativity is a very useful property even if we take it for granted in lower mathematics. We can justify associativity in much the same way as commutativity. If $x, y, z \in \mathbb{Q}$, then there exist integers m, n, p, q, r, s such that $n, q, s \neq 0$ and $x = m/n$, $y = p/q$, and $z = r/s$. Now go ahead and compute $(x + y) + z$.

$$\begin{aligned} (x + y) + z &= \left(\frac{m}{n} + \frac{p}{q}\right) + \frac{r}{s} = \frac{mq + np}{nq} + \frac{r}{s} = \frac{(mq + np)s + (nq)r}{(nq)s} \\ &= \frac{((mq)s + (np)s) + (nq)r}{nqs} = \frac{mqs + nps + nqr}{nqs} \end{aligned}$$

Similarly, compute $x + (y + z)$

$$\begin{aligned} x + (y + z) &= \frac{m}{n} + \left(\frac{p}{q} + \frac{r}{s}\right) = \frac{m}{n} + \frac{ps + qr}{qs} = \frac{m(qs) + n(ps + qr)}{n(qs)} \\ &= \frac{m(qs) + (n(ps) + n(qr))}{nqs} = \frac{mqs + nps + nqr}{nqs} \end{aligned}$$

Explain why the two results are the same.

Because we could use the distributivity of multiplication and addition on the integers and the associativity of multiplication and addition on the integers to break up and drop parentheses.

The number 0 is an additive identity in $\mathbb{Q}$. This means that if $x \in \mathbb{Q}$, then $x + 0 = x$. (Also $0 + x = x$, but since addition is commutative, this is the same thing anyway.) This is easy enough to see. Since $x \in \mathbb{Q}$, there exist integers m, n such that $n \neq 0$ and $x = m/n$. Now

$$x + 0 = \frac{m}{n} + 0 = \frac{m}{n} + 0\frac{n}{n} = \frac{m}{n} + \frac{0}{n} = \frac{m + 0}{n} = \frac{m}{n} = x.$$

Now, you might think, what is the point of all this, I've known ever since 3rd grade that adding 0 to a number does nothing. Perhaps so, but what you probably knew was that your 3rd grade teacher assured you this was so. Our goal is to understand the reasons why mathematics works the way it does, and remembering what your 3rd grade teacher told you is good, but is not really a reason why math is the way it is. In other words, the reason why 0 is an additive identity is not because your 3rd grade teacher said so.

Every rational number has an additive inverse, which is also a rational number. This means that if $x \in \mathbb{Q}$ then there exists a number $y \in \mathbb{Q}$ such that $x + y = 0$. (Also $y + x = 0$, but this is the same thing because of commutativity.) Of course, we normally denote such a number y by $-x$ and we even know, form experience at least, that $-x = \frac{-m}{n}$. So go ahead, convince me (and yourself) that $\frac{-m}{n}$ is indeed the additive inverse of x by adding it to x and showing that you get 0.

$$\frac{m}{n} + \frac{-m}{n} = \frac{m + (-m)}{n} = \frac{0}{n} = 0$$

and by commutativity of addition on rational numbers (justified above),

$$\frac{-m}{n} + \frac{m}{n} = 0$$

Now convince me that $\frac{-m}{n}$ is a rational number. (That is it satisfies the definition of a rational number.)

Since $\frac{m}{n}$ was a rational number, we know m, n are integers and $n \neq 0$. The negative of an integer is still an integer, so $-m$ is an integer. Therefore $\frac{-m}{n}$ is a rational number.

Multiplication of rational numbers Multiplication has very much the same properties of addition. I will list them and ask you to justify them. The justifications are similar to those we gave about the properties of addition, only they are a little simpler.

$\mathbb{Q}$ is closed under multiplication.

Let x, y be rational numbers. So there exist integers m, n, p, q such that $n, q \neq 0$ and $x = m/n$, $y = p/q$. Now

$$xy = \frac{m}{n}\frac{p}{q} = \frac{mp}{nq}.$$

Since m, n, p, q are integers, mp and nq are integers. For nq to be 0, n or q would have to be 0. But neither is 0, so $nq \neq 0$. Therefore xy is a rational number.

Multiplication of rational numbers is commutative.

Let x, y be rational numbers. So there exist integers m, n, p, q such that $n, q \neq 0$ and $x = m/n$, $y = p/q$.

$$xy = \frac{m}{n}\frac{p}{q} = \frac{mp}{nq}$$
$$yx = \frac{p}{q}\frac{m}{n} = \frac{pm}{qn}$$

Since multipication is commutative on the integers, $mp = pm$ and $nq = qn$. Therefore

$$xy = \frac{mp}{nq} = \frac{pm}{qn} = yx.$$

Multiplication of rational numbers is associative.

If $x, y, z \in \mathbb{Q}$, then there exist integers m, n, p, q, r, s such that $n, q, s \neq 0$ and $x = m/n$, $y = p/q$, and $z = r/s$.

$$(xy)z = \left(\frac{m}{n}\frac{p}{q}\right)\frac{r}{s} = \frac{mp}{nq}\frac{r}{s} = \frac{(mp)r}{(nq)r}$$
$$x(yz) = \frac{m}{n}\left(\frac{p}{q}\frac{r}{s}\right) = \frac{m}{n}\frac{pr}{qs} = \frac{m(pr)}{n(qr)}$$

Since multiplication is associative on the integers $(mp)r = m(pr)$ and $(nq)s = n(qs)$. So

$$(xy)z = \frac{(mp)r}{(nq)r} = \frac{m(pr)}{n(qr)} = x(yz).$$

The number 1 is a multiplicative identity in $\mathbb{Q}$.

Let x be a rational number. Then there exist integers m, n such that $n \neq 0$ and $x = m/n$. Now

$$1x = 1\frac{m}{n} = \frac{1m}{n} = \frac{m}{n} = x.$$

By commutativity of multiplication on rationals (justified above), $x1 = x$ too. Therefore 1 is a multiplicative identity.

Every nonzero rational number has a multiplicative inverse, which is also a rational number.

Let x be a nonzero rational number. Then there exist integers m, n such that $m, n \neq 0$ and $x = m/n$. First, notice that $\frac{n}{m}$ is a rational number since m, n are integers and $m \neq 0$. Now

$$\frac{m}{n}\frac{n}{m} = \frac{mn}{nm} = 1$$

because $mn = nm$. Similarly (or by commutativity of multiplication)

$$\frac{n}{m}\frac{m}{n} = 1.$$

Therefore $\frac{n}{m}$ is a rational number which is the multiplicative inverse of x.

Subtraction and division These don't have so many nice properties. It is still true that $\mathbb{Q}$ is closed under subtraction. You could justify this using a very similar argument to the one we had for closure under addition. Try it:

Let x and y be rational numbers. Then there exist some integers m, n, p, q such that $n, p \neq 0$ and $x = m/n$ and $y = p/q$.

$$\begin{aligned} x - y &= \frac{m}{n} - \frac{p}{q} \\ &= \frac{m}{n}\frac{q}{q} - \frac{p}{q}\frac{n}{n} \\ &= \frac{mq}{nq} - \frac{pn}{qn} \\ &= \frac{mq - np}{nq} \end{aligned}$$

We know m and q are integers, so their product must also be integer. Similarly, n and p are integers, so their product must also be integer. Hence $mq - np$ is the difference of two integers, and is therefore itself an integer. The denominator is nq which is a product of two integers and is therefore integer. We know that $n, q \neq 0$, and so their product cannot be 0. So $x + y$ is indeed a rational number.

Actually, a quicker way to do it is to note that $x - y = x + (-y)$ and we already showed that the additive inverse of a rational number is a rational number and the sum of two rationals is rational. So as long as $x, y \in \mathbb{Q}$, $x - y$ must be rational.

Otherwise, subtraction is not commutative or associative, does not have an identity (no 0 does not work because even though $x - 0 = x$, $0 - x \neq x$), and therefore subtractive inverses cannot exist either. Here is an easy challenge for you: show that subtraction is not commutative on $\mathbb{Q}$ by finding two rational numbers x and y such that $x - y \neq y - x$.

For example,

$$\frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6} \qquad \text{and} \qquad \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}.$$

Obviously, these are not equal.

Looking at division, it doesn't take long to notice that $\mathbb{Q}$ is not closed under division. The problem is the number 0. If we remove it, the set of nonzero rational numbers (this is usually denoted as $\mathbb{Q}^*$) is closed under division. Justify this.

Let x, y be rational numbers and $x, y \neq 0$. So there exist integers m, n, p, q such that $m, n, p, q \neq 0$ and $x = m/n$, $y = p/q$. Now

$$\frac{x}{y} = \frac{\frac{m}{n}}{\frac{p}{q}} = \frac{m}{n}\frac{q}{p} = \frac{mq}{np}.$$

Since $m, n, p, q \in \mathbb{Q}$, mq and np are integers. Also $m, n, p, q \neq 0$ shows that mq and np are not 0. Therefore the result is a nonzero rational number indeed.

Finally, division is not commutative or associative, has no identity and therefore no inverses either.