GMS 91 Homework solutions on equations and inequalities

Nov 10, 2008

Problem 8 on Homework_5: Solve the following equation.

$$|x-6| = |7x-6|$$

By definition

$$|x-6| = \begin{cases} x-6 & \text{if } x-6 \geq 0 \\ -x+6 & \text{if } x-6 < 0 \end{cases}$$

$$|7x-6| = \begin{cases} 7x-6 & \text{if } 7x-6 \geq 0 \\ -7x+6 & \text{if } 7x-6 < 0 \end{cases}$$

So we will have to consider four cases.

1. Case $x-6 \geq 0$ and $7x-6 \geq 0$.
   In this case, the equation we need to solve is
   $$x-6 = 7x-6.$$
   Here is its solution:
   $$\begin{aligned} x-6 &= 7x-6 \\ x &= 7x \\ 6x &= 0 \\ x &= 0 \end{aligned}$$
   Now the question is whether $x=0$ satisfies the two conditions $x-6 \geq 0$ and $7x-6 \geq 0$. It does not satisfy either, so we need to reject $x=0$ as a solution in this case.
2. Case $x-6 \geq 0$ and $7x-6 < 0$.
   In this case, the equation we need to solve is
   $$x-6 = -7x+6.$$
   Here is its solution:
   $$\begin{aligned} x-6 &= -7x+6 \\ 8x &= 12 \\ x &= \frac{12}{8} = \frac{3}{2} \end{aligned}$$
   Now the question is whether $x=3/2$ satisfies the two conditions $x-6 \geq 0$ and $7x-6 < 0$. It does not satisfy either, so we need to reject $x=2$ as a solution in this case.
3. Case $x-6 < 0$ and $7x-6 \geq 0$.
   In this case, the equation we need to solve is
   $$-x+6 = 7x-6.$$
   When you multiply both sides by -1, you get
   $$x-6 = -7x+6.$$
   We have already solved this equation in case 2, so we know the solution is $x=2$.
   Now the question is whether $x=3/2$ satisfies the two conditions $x-6 < 0$ and $7x-6 \geq 0$. It does, so it is indeed a solution of the original equation.
4. Case $x-6 < 0$ and $7x-6 < 0$.
   In this case, the equation we need to solve is
   $$-x+6 = -7x+6.$$

When you multiply both sides by -1, you get

$$x - 6 = 7x - 6.$$

We have already solved this equation in Case 1, so we know the solution is $x = 0$.
Now the question is whether $x = 0$ satisfies the two conditions $x - 6 < 0$ and $7x - 6 < 0$.
It does, so it is indeed a solution of the original equation.

We conclude that the equation $|x - 6| = |7x - 6|$ has two solutions, $x = 0$ and $x = 3/2$.

Just to be on the safe side, I will check my solutions:

$$|0 - 6| = |-6| = 6, \quad |7 \cdot 0 - 6| = |-6| = 6 \qquad \surd$$
$$\left|\frac{3}{2} - 6\right| = \left|\frac{3 - 12}{2}\right| = \frac{9}{2}, \quad \left|7\frac{3}{2} - 6\right| = \left|\frac{21 - 12}{2}\right| = \frac{9}{2} \qquad \surd$$

Problem 12 on Homework_5: Solve the following inequality.

$$\frac{1}{x - 8} \leq \frac{1}{x - 2}$$

We want to get rid of the fractions by multiplying both sides by $(x - 8)(x - 2)$. Since $(x - 8)(x - 2)$ could be negative, the direction of the inequality could flip. The sign of $(x - 8)(x - 2)$ depends on the signs of $x - 8$ and $x - 2$. Each could be positive or negative. So we have four cases to consider.

1. Case $x - 8 > 0$ and $x - 2 > 0$.
   In this case, $(x - 8)(x - 2) > 0$, so the direction of the inequality does not change. We get the inequality
   $$x - 2 \leq x - 8.$$
   Subtracting x from both side gives $-2 \leq -8$, which is always false, regardless of the value of x. Therefore we get no solution in this case.
2. Case $x - 8 < 0$ and $x - 2 < 0$.
   Just like in the previous case, $(x - 8)(x - 2) > 0$ leads to the inequality $-2 \leq -8$ and hence no solution.
3. Case $x - 8 > 0$ and $x - 2 < 0$.
   Notice that since $-8 < -2$, in fact $x - 8 < x - 2$. Hence no x satisfies the conditions $x - 8 > 0$ and $x - 2 < 0$. Therefore this case cannot lead to solutions.
4. Case $x - 8 < 0$ and $x - 2 > 0$.
   In this case, $(x - 8)(x - 2) < 0$, so the direction of the inequality flips. We get the inequality
   $$x - 2 \geq x - 8.$$
   Subtracting x from both side gives $-2 \geq -8$, which is always true, regardless of the value of x. Therefore any x which satisfies the conditions $x - 8 < 0$ and $x - 2 > 0$ will be a solution of the inequality. The first condition gives $x < 8$, the second gives $x > 2$.

We conclude that the solution of the original inequality is $2 < x < 8$.

Just to be on the safe side, I will check a few values in this interval to see if they satisfy the inequality:

$x = 3$**:**
$$\frac{1}{3 - 8} = -\frac{1}{5}, \quad \frac{1}{3 - 2} = 1, \quad -\frac{1}{5} \leq 1 \qquad \surd$$

$x = 5$**:**
$$\frac{1}{5 - 8} = -\frac{1}{3}, \quad \frac{1}{5 - 2} = \frac{1}{3}, \quad -\frac{1}{3} \leq \frac{1}{3} \qquad \surd$$

Actually, we see the pattern: the left hand side will be negative, while the right hand side is positive whenever $2 < x < 8$.