GMS 91 Exam 1 Solutions

Oct 7, 2009

1. (5 pts each) Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$. Be sure to explain your reasoning.

   (a) $f(n) = \pm n$

   This is not a function because it gives two values for an input.

   (b) $f(n) = \dfrac{1}{n^2 + 8}$

   This is a function. For any integer n, n^2 is a unique real number, so is $n^2 + 8$, and since $n^2 + 8 > 0$, it is always possible to take its reciprocal, which is also a unique real number.

   (c) $f(n) = \dfrac{1}{n^2 - 16}$

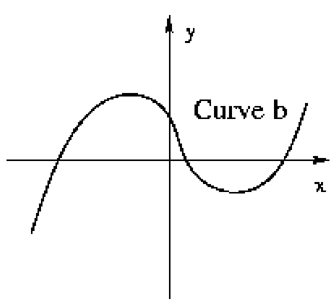
   This is not a function because it has no value for the input $n = 4$ or $n = -4$.

   (d) $f(n) = \sqrt{n^2 + 7}$

   This is a function. For any integer n, n^2 is a unique real number, so is $n^2 + 7$, and since $n^2 + 7 > 0$, it is always possible to take its square root, which is also a unique real number.

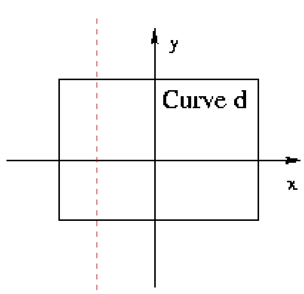
2. (2 pts each) For each of the curves below, decide if it defines y as a function of x. Explain why.

   (a)



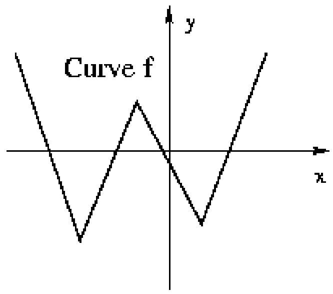
   This is the graph of a function because it passes the vertical line test: there is no vertical line which intersects it more than once.

   (b)



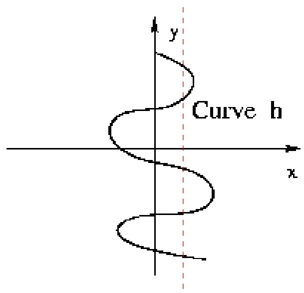
   This is not the graph of a function because it fails to pass the vertical line test: there is a vertical line which intersects it in more than one point.

   (c)



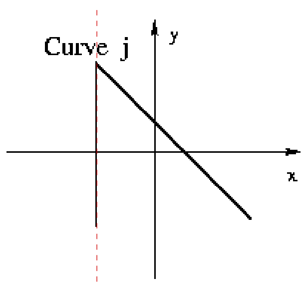
This is the graph of a function because it passes the vertical line test: there is no vertical line which intersects it more than once.

(d)



This is not the graph of a function because it fails to pass the vertical line test: there is a vertical line which intersects it in more than one point.

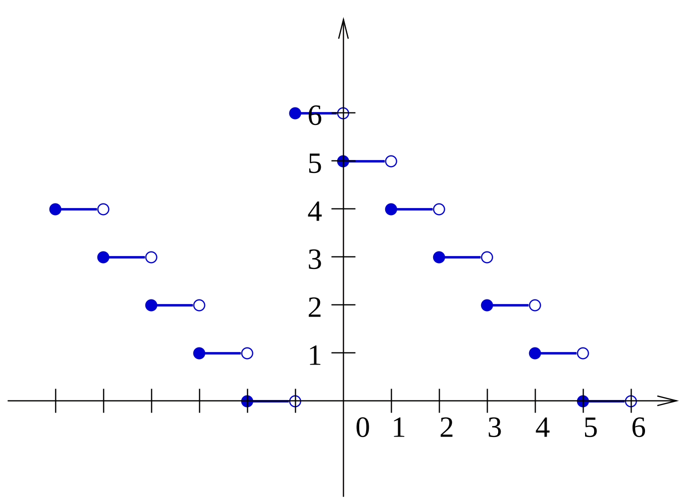
(e)



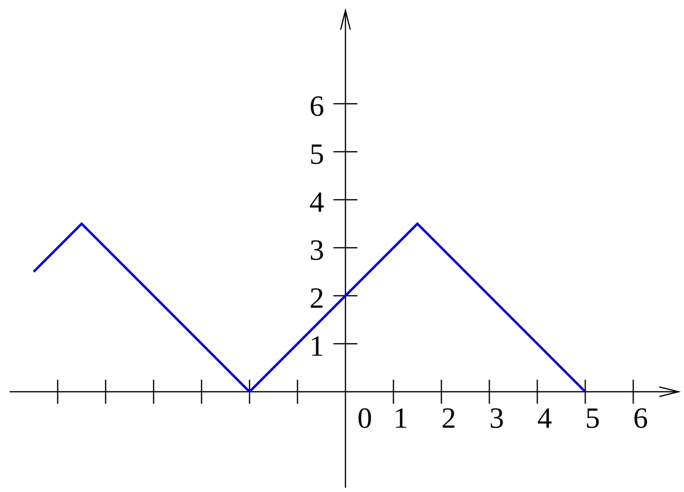
This is not the graph of a function because it fails to pass the vertical line test: there is a vertical line which intersects it in more than one point.

3. (5 pts each) Graph the following functions.

(a) The amount of time until next Friday as a function of time.



(b) The number of days to the nearest Friday (in the future or the past) as a function of time.



4. (a) (5 pts) State the Vertical Line Test.

    A 2-dimensional curve is the graph of a function if and only if no vertical line intersects it more than once.

    (b) (5 pts) Explain why the Vertical Line Test can be used to decide if a 2-dimensional curve is the graph of a function.

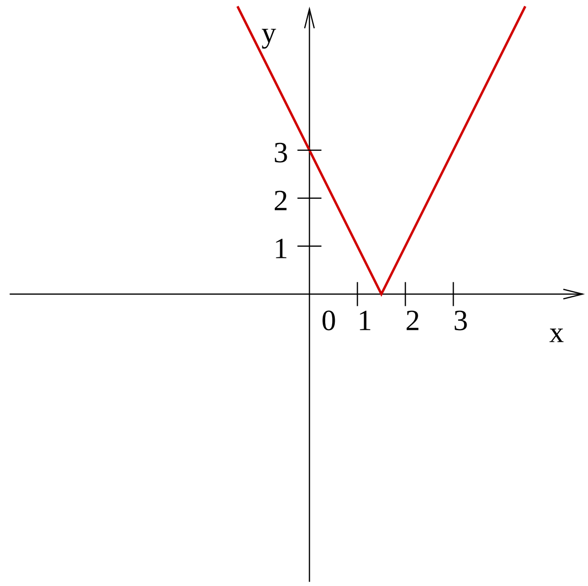
    This is because for any input, a function needs to return exactly one output. The potential inputs are on the horizontal axis. If any vertical line passes through the curve more than once, that means the input value that line passes through on the horizontal axis would correspond to more than one output value. This is not allowed for a function. On the other hand, if every vertical line intersects the curve at most once, then every point on the horizontal axis corresponds to at most one point on the curve. Some points on the horizontal axis may not correspond to any point on the curve, but this is allowed for a function. It just means that those points are not in the domain of the function.

5. (5 pts each) Graph the following functions. Pay attention to the domain.

    (a) $f : \mathbb{R} \to \mathbb{R}$ given by the rule "multiply the input by 2, subtract 3, and take the absolute value."

    This is the function $f(x) = |2x - 3|$. Since this is the absolute value of a linear function, its graph is V-shaped. You can make up a table of values to find where exactly it is. E.g.

x	-2	-1	0	1	2	3
$f(x)$	7	5	3	1	1	3

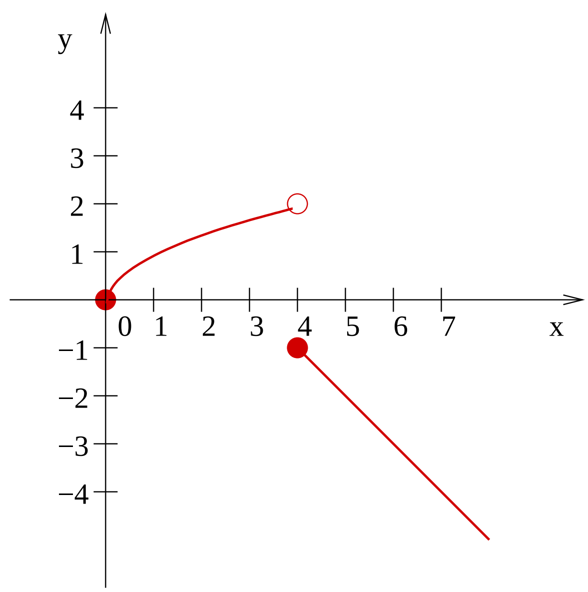


(b) $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x < 4 \\ 3 - x & \text{if } x \geq 4 \end{cases}$$

Again a table should help. The first part of the graph will look like the square root function until $x = 4$, then we have a linear formula, so the graph will look like a line. A table of values may help:

x	0	1	2	3	4	6
$f(x)$	0	1	$\sqrt{2}$	$\sqrt{3}$	-1	-3



6. (10 pts each) Let S be the set of all triangles in the plane and T be the set of all circles in the plane.

(a) Construct a rule of assignment from S to T which is a function. Explain why your rule satisfies the definition of a function.

One such rule is $f(x) =$ the inscribed circle of triangle x. Since each triangle has one and only one inscribed circle, this rule assigns to every triangle one and only one circle. Therefore it is a function.
Another possibility would be to assign each triangle to the unit circle (the circle of radius 1 centered at the origin).

(b) Construct a rule of assignment from S to T which is not a function. Explain why your rule does not satisfy the definition of a function.

One such rule is $f(x) =$ a circle with the same area as triangle x. Since there are many circles that have the same area as a specific triangle x, this rule assigns many circles to one triangle. Therefore it is not a function.

7. (15 pts) **Extra credit problem.** Let S be the set of all circles in the plane and T be the set of all triangles in the plane. If you can, construct a rule of assignment from S to T which is a function and explain why your rule satisfies the definition of a function. If you don't think this is possible, explain why not.

Finding such a rule is not easy if you want to send each circle to a different triangle. This is because it is difficult to think of one particular triangle associated with each circle. But it's easy enough to construct a rule which assigns to each circle the same triangle. E.g the triangle ABC below is a triangle in the plane. Let $f(x) = \triangle ABC$. Then $f(x)$ is a unique triangle for each circle $x \in S$, therefore f is a function $S \to T$.