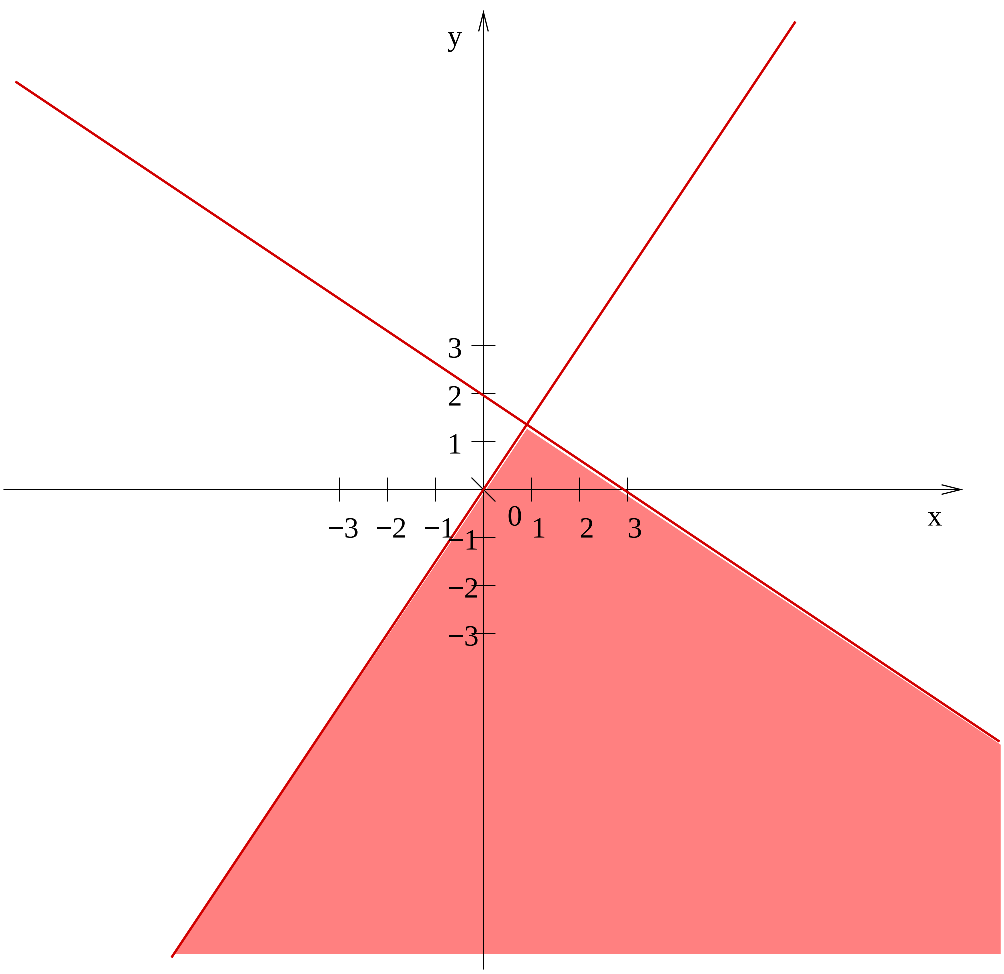
GMS 91 Final Exam Solutions

Dec 15, 2009

1. (10 pts) Graph the solution set of the system of linear inequalities

$$3x - 2y \geq 0$$
$$2x + 3y \leq 6$$

Be sure to indicate if the solution set includes its boundary.



As the solid lines indicate, the boundary is included.

2. (5 pts each)
   (a) Find x if

$$\frac{2.8^x \, 2.8^4}{2.8^{-1}} = 2.8^8.$$

$$\begin{aligned}
\frac{2.8^x\, 2.8^4}{2.8^{-1}} &= 2.8^8 \\
2.8^{x+4-(-1)} &= 2.8^8 \\
2.8^{x+5} &= 2.8^8 \\
x+5 &= 8 \\
x &= 3
\end{aligned}$$

(b) Rewrite the expression

$$\frac{\sqrt[3]{8h^9s^{-9}}}{\sqrt[3]{h^2s}}$$

in the form kh^rs^t.

$$\frac{\sqrt[3]{8h^9s^{-9}}}{\sqrt[3]{h^2s}} = \sqrt[3]{\frac{8h^9s^{-9}}{h^2s}} = \sqrt[3]{8h^7s^{-10}} = 2h^{7/3}s^{-10/3}$$

(c) Rationalize the denominator of

$$\frac{1}{11x\sqrt{5}-4y\sqrt{3}}.$$

$$\begin{aligned}
\frac{1}{11x\sqrt{5}-4y\sqrt{3}} &= \frac{1}{11x\sqrt{5}-4y\sqrt{3}}\,\frac{11x\sqrt{5}+4y\sqrt{3}}{11x\sqrt{5}+4y\sqrt{3}} = \frac{11x\sqrt{5}+4y\sqrt{3}}{(11x\sqrt{5})^2-(4y\sqrt{3})^2} \\
&= \frac{11x\sqrt{5}+4y\sqrt{3}}{121x^25-16y^23} = \frac{11x\sqrt{5}+4y\sqrt{3}}{605x^2-48y^2}
\end{aligned}$$

3. (5 pts each) Evaluate the expressions and write the result in the form $a+bi$.

(a)

$$\begin{aligned}
\frac{-4+\sqrt{-4}}{2+\sqrt{-25}} &= \frac{-4+2i}{2+5i} = \frac{-4+2i}{2+5i}\,\frac{2-5i}{2-5i} = \\
&= \frac{(-4+2i)(2-5i)}{2^2-(5i)^2} = \frac{-8+20i+4i-10i^2}{4-(-25)} = \frac{2+24i}{29} = \frac{2}{29}+\frac{24}{29}i
\end{aligned}$$

(b)

$$\begin{aligned}
\left(\frac{2+i}{-i-(4-2i)}\right)^2 &= \left(\frac{2+i}{-4+i}\right)^2 = \frac{(2+i)^2}{(-4+i)^2} \\
&= \frac{4+4i+i^2}{16-8i+i^2} = \frac{3+4i}{15-8i} \\
&= \frac{3+4i}{15-8i}\,\frac{15+8i}{15+8i} = \frac{(3+4i)(15+8i)}{15^2-(8i)^2} \\
&= \frac{45+24i+60i+32i^2}{225+64} = \frac{13}{289}+\frac{84}{289}i
\end{aligned}$$

4. (5 pts each) Let $z = 1-i$. Calculate the following:

(a) $z^2+2z+1 = (z+1)^2 = (1-i+1)^2 = (2-i)^2 = 4-4i+i^2 = 3-4i$

(b) $z^2+iz-(2+i) = (1-i)^2+i(1-i)-2-i = 1-2i+i^2+i-i^2-2-i = -1-2i$

(c) $\dfrac{(z-3)^2}{z+i} = \dfrac{(1-i-3)^2}{1-i+i} = \dfrac{(-2-i)^2}{1} = 4+4i+i^2 = 3+4i$

5. (5 pts) Suppose z varies directly with y and directly with the cube of x. If $z = 320$ when $x = 2$ and $y = 8$ then what is z when $x = 9$ and $y = 1$?

So $z = kyx^3$ for some coefficient k.

$$320 = k \cdot 8 \cdot 2^3 = 64k \implies k = 5.$$

When $x = 9$ and $y = 1$,

$$z = 5 \cdot 1 \cdot 9^3 = 3645.$$

6. (10 pts) Solve the equation $4x^2 + 12x - 7 = 0$ by completing the square.

$$\begin{aligned}
4x^2 + 12x - 7 &= 0 \\
x^2 + 3x - \frac{7}{4} &= 0 \\
\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - \frac{7}{4} &= 0 \\
\left(x + \frac{3}{2}\right)^2 &= \frac{9}{4} + \frac{7}{4} = 4 \\
x + \frac{3}{2} &= \pm 2 \\
x &= -\frac{7}{4}, \frac{1}{4}
\end{aligned}$$

7. (4 pts each) Decide if the following rules from a set to another set are functions and carefully justify your answer.

(a) Let S be the set of all cats living in the Imperial Valley and T the set of all people living in the Imperial Valley. Define $f : S \to T$ be by

$$f(x) = x\text{'s owner.}$$

This is not a function because not every cat in the Imperial Valley has an owner, hence f does not return a value for some inputs.

(b) Define $g : \mathbb{N} \to \mathbb{N}$ by

$$g(x) = \text{the sum of the digits of } x.$$

(Recall that $\mathbb{N}$ is the set of natural numbers.)

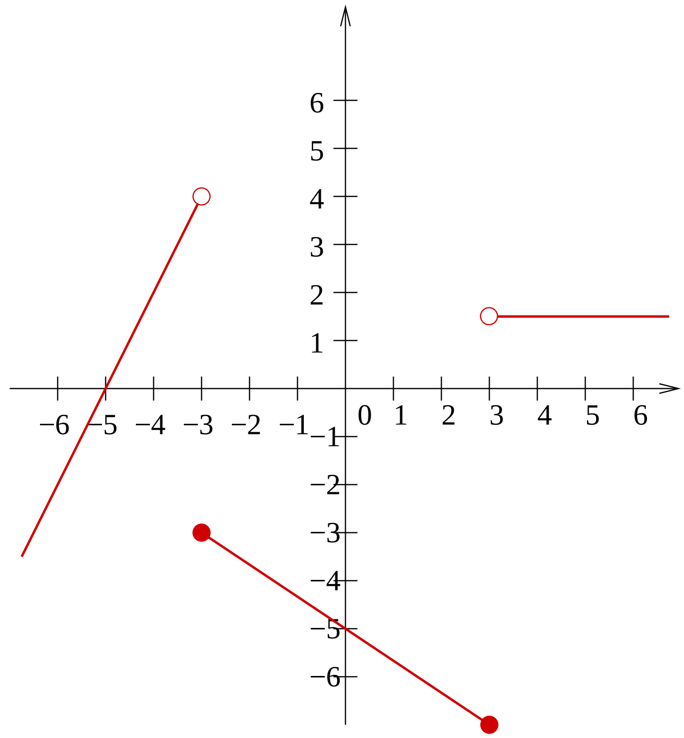
For any natural number x, the sum of the digits of x exists, is uniquely defined, and is itself a nonnegative integer. That is for any potential input x, $f(x)$ is a well-defined natural number. Therefore this is a function.

(c) Let S be the set of all triangles in the plain and $T = \mathbb{R}^+$ the of positive real numbers. Define $h : S \to T$ be by

$$h(x) = \text{the height of } x.$$

This is not a function because a triangle usually has not one but three different heights, corresponding to each of its three sides.

8. (8 pts) Given the graph of a the piecewise-defined function f from the real numbers to the real numbers, write a formula for $f(x)$.



The leftmost piece is a line with slope 2 that passes through $(-5, 0)$, hence its equation is $y = 2x + 10$. The piece in the middle is a line of slope $-2/3$ that passes through $(0, -5)$. Hence its equation is $y = -2/3x - 5$. Finally, the rightmost piece is a horizontal line whose at height $y = 3/2$. Therefore the correct formula is

$$f(x) = \begin{cases} 2x + 10 & x < -3 \\ -\frac{2}{3}x - 5 & -3 \leq x \leq 3 \\ \frac{3}{2} & x > 3 \end{cases}.$$

9. (5 pts) Find the equation of the line which passes through $(-2, 5)$ and is perpendicular to the line $3y + 4x = 1$.

The line given has slope $-4/3$. The line perpendicular to it will have slope $3/4$. Therefore its point-slope equation is

$$\begin{aligned} y - 5 &= \frac{3}{4}(x - (-2)) \\ y - 5 &= \frac{3}{4}x + \frac{3}{2} \\ y &= \frac{3}{4}x + \frac{13}{2} \end{aligned}$$

10. (10 pts) Find the solution set of the inequality

$$3x^2 - 8x + 2 > 0$$

and graph it on the real line.

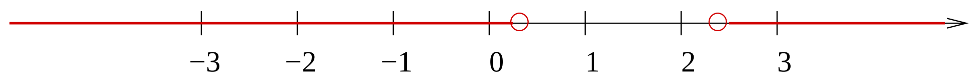
The quadratic equation $3x^2 - 8x + 2$ has roots

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 3 \cdot 2}}{6} = \frac{8 \pm \sqrt{40}}{6} = \frac{4 \pm \sqrt{10}}{3} \approx 2.39, 0.28.$$

Therefore

$$3x^2 - 8x + 2 = 3\left(x - \frac{4 + \sqrt{10}}{3}\right)\left(x - \frac{4 - \sqrt{10}}{3}\right).$$

This will be positive when both factors are positive or both are negative, that is when x is greater than both roots or x is smaller than both roots:



11. (10 pts) Solve the equation

$$\sqrt{x+1} + 2x = 8$$

Check your solution(s) by substituting them back into the equation.

$$\begin{aligned}
\sqrt{x+1} + 2x &= 8 \\
\sqrt{x+1} &= 8 - 2x \\
\sqrt{x+1}^2 &= (8 - 2x)^2 \\
x + 1 &= 64 - 32x + 4x^2 \\
4x^2 - 33x + 63 &= 0
\end{aligned}$$

This is a quadratic equation in x and can be solved by the quadratic formula:

$$x = \frac{33 \pm \sqrt{33^3 - 4 \cdot 4 \cdot 63}}{8} = \frac{33 \pm \sqrt{81}}{8} = \frac{33 \pm 9}{8} = 3, \frac{21}{4}.$$

We need to check these since we could have a false root:

$$\sqrt{3+1} + 2 \cdot 3 = \sqrt{4} + 6 = 8 \qquad \checkmark$$

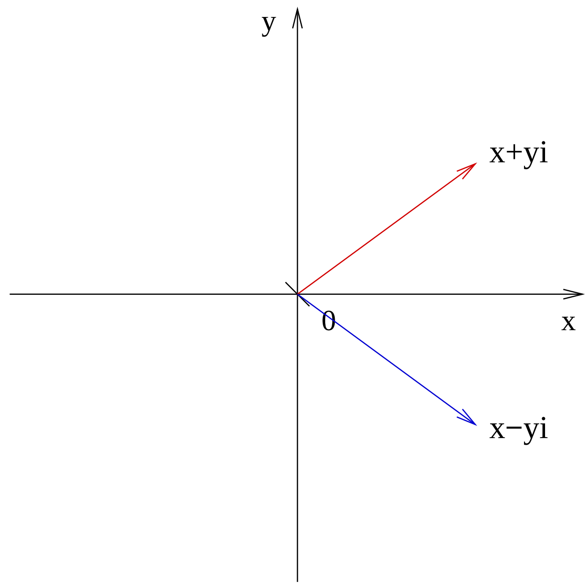
$$\sqrt{\frac{21}{4} + 1} + 2\,\frac{21}{4} = \sqrt{\frac{25}{4}} + \frac{21}{2} = \frac{5}{2} + \frac{21}{2} = 13 \qquad \times$$

So $x = 3$ is the only solution and $x = 21/4$ is a false root.

12. (10 pts each) Let $z = x + yi$.
   (a) Prove that $|z| = |\overline{z}|$ by an algebraic argument.

$$|\overline{z}| = |\overline{x + yi}| = |x - yi| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |x + yi| = |z|$$

   (b) Plot z and $\overline{z}$ in the complex plane. Give a geometric argument why $|z|$ and $|\overline{z}|$ must be equal.



The absolute value of a number is its distance from the origin. As the plot above shows, z and $\overline{z}$ are mirror images of each other reflected across the horizontal axis. Because of the mirror symmetry, z and $\overline{z}$ are the same distance from the origin, therefore $|z| = |\overline{z}|$.

13. (10 pts) **Extra credit problem.** Due to budget cuts, Santa Claus and his elves are required to observe two days of furloughs each month this year. On a normal workday, when Santa and the elves are all working, it takes them 10 hours to meet the daily production target of toys. On a day when the elves are furloughed, Santa, working alone, takes 4.5 hours longer to meet the daily target than the elves take, working by themselves, on a day when Santa is furloughed. How long does Santa take to make the toys when he is working by himself?

Santa at the end of the elves' furlough day

Let x be the number of hours the elves take working without Santa to meet the daily production target. Then Santa, working by himself, will take $x + 4.5$ hours to meet the target. So in one hour, the elves by themselves, meet $1/x$ of the daily target. Santa working by himself for an hour will meet $1/(x + 4.5)$ of the daily target. Working together for 10 hours, Santa and the elves meet $10(1/x + 1(x + 4.5))$ of the daily target. But this is 100% of the daily target. So

$$
\begin{aligned}
10\left(\frac{1}{x} + \frac{1}{x+4.5}\right) &= 1 \\
\frac{10}{x} + \frac{10}{x+4.5} &= 1 \\
10(x+4.5) + 10x &= x(x+4.5) \\
10x + 45 + 10x &= x^2 + 4.5x \\
x^2 - 15.5x - 45 &= 0
\end{aligned}
$$

We can solve this equation using the quadratic formula:

$$x = \frac{15.5 \pm \sqrt{15.5^2 + 4 \cdot 45}}{2} = \frac{312 \pm \sqrt{\frac{31^2}{4} + 180}}{2} = \frac{312 \pm \frac{41}{2}}{2} = -\frac{5}{2}, 18$$

Clearly, $-5/2$ does not make sense as a solution. So we can conclude that Santa, working alone, will take $18 + 4.5 = 22.5$ hours to meet the daily production target. Poor Santa!