Name: ______________________

GMS 91 Worksheet on complex numbers

Nov 18, 2009

This worksheet is due by 8 PM on Wed, 12/2. You can do it individually, or in groups of two. But no more than two of you should form a group.

In class, we talked about the conjugate of a complex number. We said that if $z = x + yi$ (where x and y are real numbers), then the conjugate of z is $\overline{z} = x - yi$. For example $\overline{3+4i} = 3 - 4i$. For starters, find the following conjugates.

1. $\overline{-5+6i}$

2. $\overline{-7-10i}$

3. $\overline{5i}$

4. $\overline{9}$

5. $\overline{0}$

Now, plot these numbers along with their conjugates in the complex plane.

We also talked about the absolute value (magnitude, length) of a complex number. Let z be the complex number $x + yi$, we gave two ways of computing the absolute value:

1. $|z| = \sqrt{x^2 + y^2}$
2. $|z| = \sqrt{z\bar{z}} = \sqrt{(x + yi)(x - yi)}$

Find the absolute values below using both methods.

1. $|-5 + 6i|$

2. $|-7 - 10i|$

3. $|5i|$

4. $|9|$

5. $|0|$

Do the numbers you get make sense given your plot of these numbers in the complex plane?

You will now discover some properties of the conjugate. Compute the following.

1. $(\overline{3-4i}) + (\overline{8+3i})$ and $\overline{(3-4i)+(8+3i)}$

2. $(\overline{1+2i}) + (\overline{7-2i})$ and $\overline{(1+2i)+(7-2i)}$

3. $(\overline{5-6i}) + (\overline{10+i})$ and $\overline{(5-6i)+(10+i)}$

What do you notice?

Let $w = a + bi$ and $z = x + yi$ be complex numbers.

1. Find $\overline{w} + \overline{z}$ and $\overline{w+z}$.

2. Does your result show you a general principle about the conjugate?

3. Can you give a geometric explanation of this?

Now, try this with subtraction. Compute the following.

1. $(\overline{3-4i}) - (\overline{8+3i})$ and $\overline{(3-4i)-(8+3i)}$

2. $(\overline{1+2i}) - (\overline{7-2i})$ and $\overline{(1+2i)-(7-2i)}$

3. $(\overline{5-6i}) - (\overline{10+i})$ and $\overline{(5-6i)-(10+i)}$

What do you notice?

Let $w = a + bi$ and $z = x + yi$ be complex numbers.

1. Find $\overline{w} - \overline{z}$ and $\overline{w - z}$.

2. Did you just discover another general principle of the conjugate?

3. Can you give a geometric explanation of this?

Actually, this even works with just negation. Try these.

1. $\overline{-(3-4i)}$ and $-(\overline{3-4i})$

2. $\overline{-(1+2i)}$ and $-(\overline{1+2i})$

3. $\overline{-(5-6i)}$ and $-(\overline{5-6i})$

4. $\overline{-(x+yi)}$ and $-(\overline{x+yi})$

Okay. What about multiplication? Here we go. Compute the following.

1. $(\overline{3-4i})(\overline{8+3i})$ and $\overline{(3-4i)(8+3i)}$

2. $(\overline{1+2i})(\overline{7-2i})$ and $\overline{(1+2i)(7-2i)}$

3. $(\overline{5-6i})(\overline{10+i})$ and $\overline{(5-6i)(10+i)}$

What do you notice?

Let $w = a + bi$ and $z = x + yi$ be complex numbers.

1. Find $\overline{w}\,\overline{z}$ and $\overline{(wz)}$.

2. Did you discover yet another general property of the conjugate?

3. Can you give a geometric explanation of this? How about a simple algebraic explanation?

Let's see what happens when you conjugate twice. So $\overline{\overline{z}}$ means the conjugate of the conjugate of z. Try these.

1. $\overline{\overline{3 - 4i}}$

2. $\overline{\overline{1 + 2i}}$

3. $\overline{\overline{5 - 6i}}$

4. $\overline{\overline{x + yi}}$

Yet another principle, isn't it?

1. Is there a simple geometric explanation to why this is so?

2. What do you think happens if you conjugate a complex number three times? Four times?

Back to the absolute value. Compute the following.

1. $|3-4i||8+3i|$ and $|(3-4i)(8+3i)|$

2. $|1+2i||7-2i|$ and $|(1+2i)(7-2i)|$

3. $|5-6i||10+i|$ and $|(5-6i)(10+i)|$

What seems to be the general principle?

Justify it by letting $w = a + bi$, $z = x + yi$ and finding $|w||z|$ and $|wz|$.

Hopefully that worked out for you. What a messy computation! There is an easier way to do it, which also lets you see better why $|w||z| = |wz|$. Use the fact that $|w| = \sqrt{w\overline{w}}$ and $|z| = \sqrt{z\overline{z}}$. So what is $|wz|$ then?

Can you tell it is the same thing as $|w||z|$?

Do you think that if w and z are complex numbers (and $z \neq 0$), then

1. $\overline{\left(\frac{w}{z}\right)} = \frac{\overline{w}}{\overline{z}}$ and

2. $\left|\frac{w}{z}\right| = \frac{|w|}{|z|}$?

Make up a few examples for yourself and try them.

What do you observe?

Can you state general principles and justify them?

Let's see if the same thing works for addition and subtraction. Find the following.

1. $|3-4i|+|8+3i|$ and $|(3-4i)+(8+3i)|$

2. $|1+2i|+|7-2i|$ and $|(1+2i)+(7-2i)|$

3. $|3-4i|-|8+3i|$ and $|(3-4i)-(8+3i)|$

4. $|1+2i|-|7-2i|$ and $|(1+2i)-(7-2i)|$

What do you observe?

There are actually some relations between $|w|, |z|$ and $|w+z|$ and $|w-z|$, but they are not as simple as equality. For now, it's enough to know that $|w|+|z| \neq |w+z|$ and $|w|-|z| \neq |w-z|$ in general.

Here is one more property of the absolute value. Do the following.

1. Let $z = 3 - 4i$ and find $|z|$ and $|-z|$.

2. Let $z = 1 + 2i$ and find $|z|$ and $|-z|$.

3. Let $z = 5 - 6i$ and find $|z|$ and $|-z|$.

4. Let $z = x + yi$ and find $|z|$ and $|-z|$.

What do you see?

Can you see a nice geometric explanation of this principle?

Alright, you are done. You have just done what a mathematician would have done. You played with concrete examples to conjecture (that's a fancy word for guess) general principles. Then you found general arguments to show that those general principles must hold. You started with a basic understanding of conjugate and absolute value and you developed a more sophisticated understanding of their properties.