GMS 91 Exam 2 Solutions

Nov 10, 2010

11. Let f be a relation from $\mathbb{R}$ to $\mathbb{R}$.
   (a) (4 pts) State the Vertical Line Test for deciding whether f is a function.

   The relation f is a function if and only if the every vertical line intersects its graph at most once.

   (b) (6 pts) Explain why the Vertical Line Test can be used to tell if f is a function.

   A relation f is a function if every element in its domain appears exactly once as the first coordinate in the ordered pairs of f. When f is a relation from $\mathbb{R}$ to $\mathbb{R}$, this is exactly what the Vertical Line Test checks. If any vertical line crosses the graph of f more than once, then there are at least two points on the graph of f with the same x-coordinate. These two points correspond to two ordered pairs with the same first coordinates.

12. (6 pts) Find the largest possible domain within the set of real numbers of a function given by the formula

$$g(y) = \sqrt{\frac{y-3}{|y|-3}}.$$

Express your answer in interval notation. Be sure to carefully explain your work.

We need to make sure that $|y|-3 \neq 0$ (so we don't divide by 0) and $\frac{y-3}{|y|-3} \geq 0$ (so we don't take the square root of a negative number). So

$$\begin{aligned} |y|-3 &\neq 0 \\ |y| &\neq 3 \\ y &\neq \pm 3 \end{aligned}$$

To solve $\frac{y-3}{|y|-3}$, it is tempting to multiply by $|y|-3$. But we don't know if $|y|-3$ is positive or negative. So we don't know whether the direction of the inequality should flip or remain the same. We will need to consider two cases:

Case $|y|-3>0$: In this case,

$$\begin{aligned} |y|-3 &> 0 \\ |y| &> 3 \end{aligned}$$

which is the case if $y>3$ or $y<-3$.

$$\begin{aligned} \frac{y-3}{|y|-3} &\geq 0 \\ \frac{y-3}{|y|-3}(|y|-3) &\geq 0(|y|-3) \\ y-3 &\geq 0 \\ y &\geq 3 \end{aligned}$$

Both $y \geq 3$, and $y>3$ or $y<-3$ have to hold in this case. So this case yields $y>3$. In interval notation, $y \in (3,\infty)$.

Case $|y|-3<0$: In this case,

$$\begin{aligned} |y|-3 &< 0 \\ |y| &< 3 \end{aligned}$$

which is the case if $-3<y<3$.

$$\begin{aligned} \frac{y-3}{|y|-3} &\geq 0 \\ \frac{y-3}{|y|-3}(|y|-3) &\leq 0(|y|-3) \\ y-3 &\leq 0 \\ y &\leq 3 \end{aligned}$$

Both $y\leq 3$ and $-3<y<3$ have to hold in this case. So this case yields $-3<y<3$. In interval notation $y\in(-3,3)$.

That is the domain of g is $(-3,3)\cup(3,\infty)$.

13. (4 pts) You watched three short movies about mathematics in class on Nov 1. The third movie was about great advances in the history of math. Name and briefly describe one of the great advances presented in the movie.

The movie mentions quite a few such advances. Some of them are Euclid's Elements, the Pythagorean Theorem, the discovery of irrational numbers, the definition and computation of π, the invention of trigonometry, coordinate geometry, calculus, etc.

14. (a) (8 pts) Use substitution to find all real solutions of

$$\begin{aligned} 3x-2y+7z+4 &= 0 \\ 2x-y+5z &= 1 \\ 5x-11y &= 14-49z \end{aligned}$$

First, use the second equation to express y as

$$y=2x+5z-1.$$

Now, substitute this into the first equation:

$$\begin{aligned} 3x-2(2x+5z-1)+7z+4 &= 0 \\ 3x-4x-10z+2+7z+4 &= 0 \\ -x-3z+6 &= 0 \end{aligned}$$

and into the third equation:

$$\begin{aligned} 5x-11(2x+5z-1) &= 14-49z \\ 5x-22x-55z+11 &= 14-49z \\ -17x-55z+11-14+49z &= 0 \\ -17x-6z-3 &= 0. \end{aligned}$$

So now we are solving

$$\begin{aligned} -x-3z+6 &= 0 \\ -17x-6z-3 &= 0. \end{aligned}$$

Use the first of these equations to express $x = 6 - 3z$ and substitute this into the second equation:

$$\begin{aligned}
-17(6-3z) - 6z - 3 &= 0 \\
-102 + 51z - 6z - 3 &= 0 \\
45z - 105 &= 0 \\
45z &= 105 \\
z &= \frac{105}{45} = \frac{7}{3}
\end{aligned}$$

So

$$\begin{aligned}
x &= 6 - 3z = 6 - 3\frac{7}{3} = -1 \\
y &= 2x + 5z - 1 = 2(-1) + 5\frac{7}{3} - 1 = \frac{26}{3}
\end{aligned}$$

(b) (2 pts) Check your solution(s) to part (a).

$$\begin{aligned}
2(-1) - \frac{26}{3} + 5\frac{7}{3} &= \frac{-6 - 26 + 35}{3} = \frac{3}{3} = 1\surd \\
3(-1) - 2\frac{26}{3} + 7\frac{7}{3} + 4 &= \frac{-9 - 52 + 49 + 12}{3} = 0\surd \\
5(-1) - 11\frac{26}{3} &= \frac{-15 - 286}{3} = -\frac{301}{3} \\
14 - 49\frac{7}{3} &= \frac{42 - 343}{3} = -\frac{301}{3}\surd
\end{aligned}$$

15. (5 pts each) **Extra credit problem.** Let S be the set of all triangles in the plane. Let T be the set of all points in the plane.

(a) Define the relation R from S to T as

$$R = \{(x, y) \mid y \text{ is a vertex of } x\}.$$

Is R a function? Why or why not?

R is not a function. Every triangle has three vertices. So every triangle x will appear three times as the first coordinate in an ordered pair (x, y).

(b) Define the relation Q from T to S as

$$Q = \{(x, y) \mid x \text{ is a vertex of } y\}.$$

Is Q a function? Why or why not?

Q is not a function either. Every point in the plane can be the vertex of infinitely many triangles. So every point x will appear infinitely many times as the first coordinate of an ordered pair (x, y).