GMS 91 Final Exam Solutions

Dec 15, 2010

See the solutions of problems 1–16 in HLS.

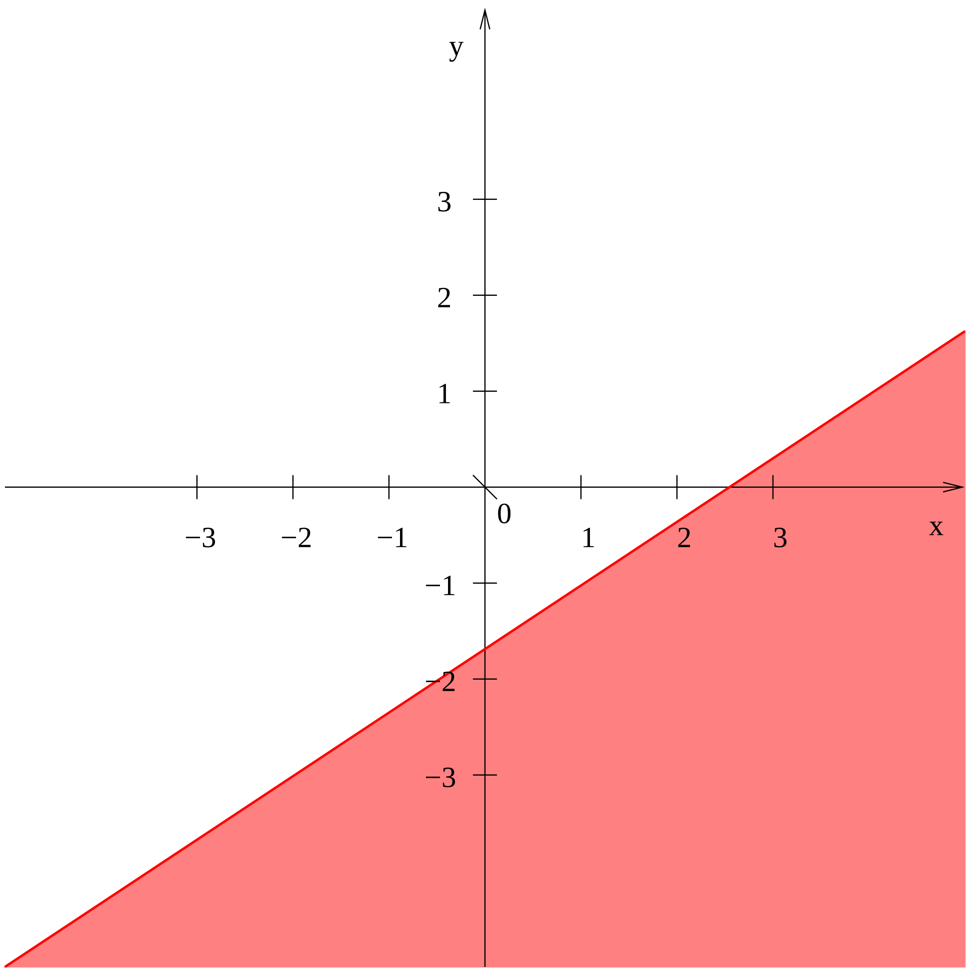
17. (10 pts) Graph the solutions of the linear inequality

$$2x - y > 5 + 2y.$$

First, convert the inequality into the equation $2x - y = 5 + 2y$. Now

$$2x - y = 5 + 2y \implies 3y = 2x - 5 \implies y = \frac{2}{3}x - \frac{5}{3}.$$

This is the equation of a line which passes through $(0, -5/3)$ and a has a slope of $2/3$. We can now graph this line. You can now easily verify that $(0, 0)$ is not a solution of this inequality, hence its solution set is the half-plane which does not include the origin.



18. (10 pts) Solve the equation

$$7x - 8 = 2(x + 3)$$

for the real number x. Justify each step by referring to an appropriate algebraic property. Be sure not to skip or combine steps.

$$
\begin{aligned}
7x - 8 &= 2(x+3) && \\
7x - 8 &= 2x + 6 && \text{distributivity of multiplication over addition} \\
(7x-8)+8 &= (2x+6)+8 && \text{addition property of equality} \\
7x + (-8+8) &= 2x + (6+8) && \text{associativity of addition} \\
7x + (-8+8) &= 2x + 14 && \text{arithmetic} \\
7x + 0 &= 2x + 14 && \text{inverse property of addition} \\
7x &= 2x + 14 && \text{identity property of addition} \\
-2x + 7x &= -2x + (2x+14) && \text{addition property of equality} \\
-2x + 7x &= (-2x+2x) + 14 && \text{associativity of addition} \\
-2x + 7x &= (-2+2)x + 14 && \text{distributivity of multiplication over addition} \\
-2x + 7x &= 0x + 14 && \text{inverse property of addition} \\
-2x + 7x &= 0 + 14 && \text{zero property of multiplication} \\
-2x + 7x &= 14 && \text{identity property of addition} \\
(-2+7)x &= 14 && \text{distributivity of multiplication over addition} \\
5x &= 14 && \text{arithmetic} \\
\frac{1}{5}(5x) &= \frac{1}{5}14 && \text{multiplication property of equality} \\
\frac{1}{5}(5x) &= \frac{14}{5} && \text{arithmetic} \\
\left(\frac{1}{5}5\right)x &= \frac{14}{5} && \text{associativity of multiplication} \\
1x &= \frac{14}{5} && \text{inverse property of multiplication} \\
x &= \frac{14}{5} && \text{identity property of multiplication}
\end{aligned}
$$

19. (a) (2 pts) State the definition of a relation.

A relation R from a set S to a set T is a set of ordered pairs (s, t) where s is an element of S and t is an element of T.

(b) (2 pts) State the definition of a function.

A function f from a set S to a set T is a relation from S to T such that each element of S appears exactly once as the first coordinate of an ordered pair.

(c) (4 pts) Let S be the set of all 9-digit numbers and T the set of all people in the world. Let $f : S \to T$ be defined by

$$f(x) = \text{the person whose social security number is } x.$$

Is f a relation? Why or why not? Is f a function? Why or why not?

Yes, f is certainly a relation since it consists of ordered pairs (x, y) such that x is a 9-digit number (hence an element of S) and y is a person (hence an element of T). But f is not a function because there are 9-digit numbers that are not anybody's social security numbers. This is quite clear, since there are a billion 9-digit numbers, but there are only about 300 million people in the USA. Such numbers do not appear as the first coordinate of an ordered pair in f. This is why f is not function.

(d) (7 pts) Find the largest possible domain within the real numbers of the function

$$f(x) = \sqrt{\frac{3}{2x+11}}.$$

The expression can be evaluated as long as the quantity under the square root is a nonnegative number. This is the case whenever $2x + 11 > 0$. So

$$\begin{aligned} 2x + 11 &> 0 \\ 2x &> -11 \\ x &> -\frac{11}{2} \end{aligned}$$

So the largest possible domain of f is $\{x|x > -11/2\} = [-11/2, \infty)$.

20. (15 pts) Solve the radical equation

$$\sqrt{3x+10} = 1 + \sqrt{7-x}$$

where x is a real number.

$$\begin{aligned} \sqrt{3x+10} &= 1 + \sqrt{7-x} \\ (\sqrt{3x+10})^2 &= (1+\sqrt{7-x})^2 \\ 3x + 10 &= 1 + 2\sqrt{7-x} + (7-x) \\ 3x + 10 &= 8 - x + 2\sqrt{7-x} \\ 3x + 10 - 8 + x &= 2\sqrt{7-x} \\ 4x + 2 &= 2\sqrt{7-x} \\ 2x + 1 &= \sqrt{7-x} \\ (2x+1)^2 &= (\sqrt{7-x})^2 \\ 4x^2 + 4x + 1 &= 7 - x \\ 4x^2 + 4x + 1 - 7 + x &= 0 \\ 4x^2 + 5x - 6 &= 0 \end{aligned}$$

We can solve this polynomial equation by factoring $4x^2 + 5x - 6$. One way to do this is to use the AC-method. We are looking for two integers whose sum is $B = 5$ and whose product is $AC = 4(-6) = -24$. It is easy enough to see that 8 and -3 will work. So

$$0 = 4x^2 + 5x - 6 = 4x^2 + 8x - 3x - 6 = 4x(x+2) - 3(x+2) = (4x-3)(x+2).$$

A product of two numbers is 0 when at least one of the factors is 0, so the two candidates for the solution are $4x + 3 = 0$ or $x - 2 = 0$, i.e. $x = 3/4$ or $x = -2$.

Because we squared the equation twice, we may have introduced false roots. So we need to check these potential roots:

$$\sqrt{3\frac{3}{4} + 10} = \sqrt{\frac{9+40}{4}} = \sqrt{\frac{49}{4}} = \frac{7}{2}$$

$$1+\sqrt{7-\frac{3}{4}}=1+\sqrt{\frac{28-3}{4}}=1+\sqrt{\frac{25}{4}}=1+\frac{5}{2}=\frac{7}{2} \qquad \checkmark$$
$$\sqrt{3(-2)+10}=\sqrt{4}=2$$
$$1+\sqrt{7-(-2)}=1+\sqrt{9}=1+3=4 \qquad \times$$

So the only solution is $x=3/4$.

21. **Extra credit problem.** We learned that if $z=x+yi$ is a complex number then its complex conjugate is $\overline{z}=x-yi$.

(a) (5 pts) Show that if x and w are complex numbers, then

$$\overline{(z+w)}=\overline{z}+\overline{w}.$$

(Hint: Start by letting $z=x+yi$ and $w=a+bi$.

Let $z=x+yi$ and $w=a+bi$. Then

$$\begin{aligned}\overline{z}+\overline{w}&=\overline{(x+yi)}+\overline{(s+ti)}=(x-yi)+(s-ti)\\&=x+s-(y+t)i=\overline{(x+s+(y+t)i)}=\overline{(x+yi+s+ti)}=\overline{z+w}\end{aligned}$$

(b) (10 pts) Show that if x and $w\neq 0$ are complex numbers, then

$$\overline{\left(\frac{z}{w}\right)}=\frac{\overline{z}}{\overline{w}}.$$

Let $z=x+yi$ and $w=a+bi$. Then

$$\begin{aligned}\overline{\left(\frac{z}{w}\right)}&=\overline{\left(\frac{x+yi}{a+bi}\right)}\\&=\overline{\left(\frac{x+yi}{a+bi}\,\frac{a-bi}{a-bi}\right)}\\&=\overline{\left(\frac{(x+yi)(a-bi)}{(a+bi)(a-bi)}\right)}\\&=\overline{\left(\frac{xa-xbi+ayi-ybi^2}{a^2-(bi)^2}\right)}\\&=\overline{\left(\frac{xa+yb+(-xb+ay)i}{a^2+b^2}\right)}\end{aligned}$$

Now, notice that a^2+b^2 is a real number, so we can simply divide the real and the imaginary parts of $xa+yb+(-xb+ay)i$ by a^2+b^2 as

$$\frac{xa+yb+(-xb+ay)i}{a^2+b^2}=\frac{xa+yb}{a^2+b^2}+\frac{-xb+ay}{a^2+b^2}i.$$

So

$$\begin{aligned}\overline{\left(\frac{xa+yb+(-xb+ay)i}{a^2+b^2}\right)}&=\overline{\left(\frac{xa+yb}{a^2+b^2}+\frac{-xb+ay}{a^2+b^2}i\right)}\\&=\frac{xa+yb}{a^2+b^2}-\frac{-xb+ay}{a^2+b^2}i\\&=\frac{xa+yb}{a^2+b^2}+\frac{xb-ay}{a^2+b^2}i\end{aligned}$$

We will now do it the other way:

$$\begin{aligned}
\frac{\overline{z}}{\overline{w}} &= \frac{\overline{x+yi}}{\overline{a+bi}} \\
&= \frac{x-yi}{a-bi} \\
&= \frac{x-yi}{a-bi}\,\frac{a+bi}{a+bi} \\
&= \frac{(x-yi)(a+bi)}{(a-bi)(a+bi)} \\
&= \frac{xa+xbi-ayi-ybi^2}{a^2-(bi)^2} \\
&= \frac{xa+yb+(xb-ay)i}{a^2+b^2}
\end{aligned}$$

Once again, a^2+b^2 is a real number, so we can simply divide the real and the imaginary parts of $xa+yb+(xb-ay)i$ by a^2+b^2 as

$$\frac{xa+yb+(xb-ay)i}{a^2+b^2} = \frac{xa+yb}{a^2+b^2} + \frac{xb-ay}{a^2+b^2}i.$$

Notice that this is the same as the result we got when we evaluated $\overline{\left(\frac{z}{w}\right)}$ above. So

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}.$$