GMS 91 Final Exam Solutions

May 11, 2011

See solutions to problems 1–20 in Hawkes Learning System.

21. (10 pts) Let x be any real number. Prove that $0 \cdot x = 0$.

    You can find a proof on the solution set to the first exam this semester.

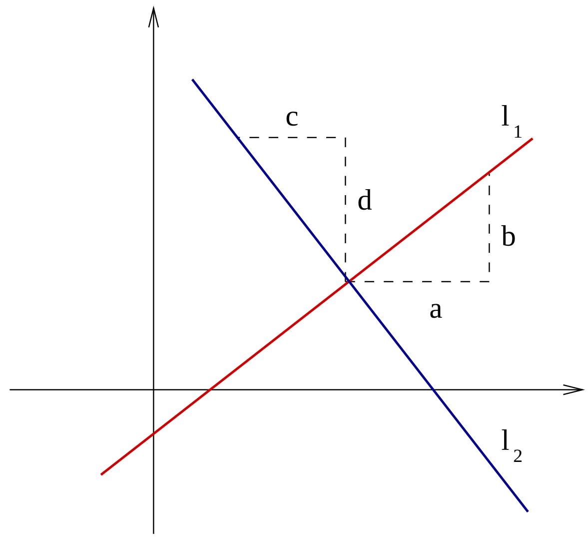
22. (10 pts) Solve the equation

$$5 - 3x = 2 + x$$

for the real number x. Justify each step by referring to an appropriate algebraic property. Be sure not to skip or combine steps.

$5 - 3x = 2 + x$	
$5 + (-3x) = 2 + x$	definition of subtraction
$(5 + (-3x)) + 3x = (2 + x) + 3x$	addition property of equality
$5 + (-3x + 3x) = (2 + x) + 3x$	associativity of addition
$5 + 0 = (2 + x) + 3x$	inverse property of addition
$5 = (2 + x) + 3x$	identity property of addition
$5 = 2 + (x + 3x)$	associativity of addition
$5 = 2 + (1 + 3)x$	distributivity of multiplication over addition
$5 = 2 + 4x$	arithmetic
$-2 + 5 = -2 + (2 + 4x)$	addition property of equality
$3 = -2 + (2 + 4x)$	arithmetic
$3 = (-2 + 2) + 4x$	associativity of addition
$3 = 0 + 4x$	inverse property of addition
$3 = 4x$	identity property of addition
$\frac{1}{4}3 = \frac{1}{4}(4x)$	multiplication property of equality
$\frac{3}{4} = \frac{1}{4}(4x)$	arithmetic
$\frac{3}{4} = \left(\frac{1}{4}4\right)x$	associativity of multiplication
$\frac{3}{4} = 1x$	inverse property of multiplication
$\frac{3}{4} = x$	identity property of multiplication

23. (10 pts) Let l_1 and l_2 be lines with slopes m_1 and m_2 respectively. We learned this semester that if $m_1 m_2 = -1$ then the two lines are perpendicular. Justify this. (Hint: Draw a picture which compares runs and rises.)



In the picture above, the legs of the two right triangles are the rises and runs of l_1 and l_2. So

$$m_1 = \frac{b}{a}, \qquad m_2 = \frac{-d}{c}.$$

(It is $-d$ because that side points downward.) So

$$-1 = m_1 m_2 = \frac{b}{a}\frac{-d}{c} = \frac{-bd}{ac}.$$

But we can always choose c, the run of l_2, to be the same as b, the rise of l_1. Then

$$-1 = \frac{-bd}{ac} = \frac{-bd}{ab} = \frac{-d}{a} \implies d = a.$$

So the two right triangles are congruent because their corresponding legs are equal. But a and d are perpendicular (because one is horizontal and the other is vertical), so l_1 and l_2 must also be perpendicular.

24. (10 pts) Find the largest possible domain within the real numbers which makes

$$h(y) = \sqrt{|3 - 2y| - y}$$

a function. Give your answer in interval notation.

Because of the square root, $|3 - 2y| - y$ must not be negative. Any nonnegative value is fine though because any nonnegative real number has a square root that is a real number. So let us figure out when $|3 - 2y| - y \geq 0$. Because of the absolute value, we have two cases:

(a) Case $|3 - 2y| = 3 - 2y$. This happens when $3 - 2y \geq 0$. That is $3/2 \geq y$.

$$\begin{aligned} (3 - 2y) - y &\geq 0 \\ 3 - 3y &\geq 0 \\ 3 &\geq 3y \\ 1 &\geq y \end{aligned}$$

This is a stronger condition than $3/2 \geq y$, so this case yields $1 \geq y$ or $(-\infty, 1]$ as a set of solutions.

(b) Case $|3-2y| = -(3-2y)$. This happens when $3-2y<0$. That is $3/2<y$.

$$\begin{aligned} -(3-2y)-y &\geq 0 \\ y-3 &\geq 0 \\ y &\geq 3 \end{aligned}$$

This is a stronger condition than $3/2<y$, so this case yields $y \geq 3$ or $[3,\infty)$ as a set of solutions

So the largest possible domain is $(-\infty,1]\cup[3,\infty)$.

25. (5 pts each)

(a) Factor the polynomial $6x^3+7x^2-20x$ completely.

First of all,

$$6x^3+7x^2-20x = x(6x^2+7x-20).$$

Now, we can use the AC-method to factor $6x^2+7x-20$. So we are looking for two numbers whose product is $6(-20)=-120$ and whose sum is 7. It is easy to find that 15 and -8 will do the job. Now

$$\begin{aligned} 6x^2+7x-20 &= 6x^2+15x-8x-20 \\ &= 3x(2x+5)-4(2x+5) \\ &= (3x-4)(2x+5) \end{aligned}$$

Therefore

$$6x^3+7x^2-20x = x(3x-4)(2x+5).$$

Since the factors are linear and have no nontrivial common divisors, this is a complete factoring.

(b) Use the factorization from part (a) to find the solutions of the polynomial equation $6x^3+7x^2-20x=0$.

Since a product of real numbers is 0 exactly when at least one of the factors is 0,

$$0 = 6x^3+7x^2-20x = x(3x-4)(2x+5) \iff x=0 \text{ or } 3x-4=0 \text{ or } 2x+5=0.$$

So the solutions are

$$x=0, x=\frac{4}{3}, x=-\frac{5}{2}.$$

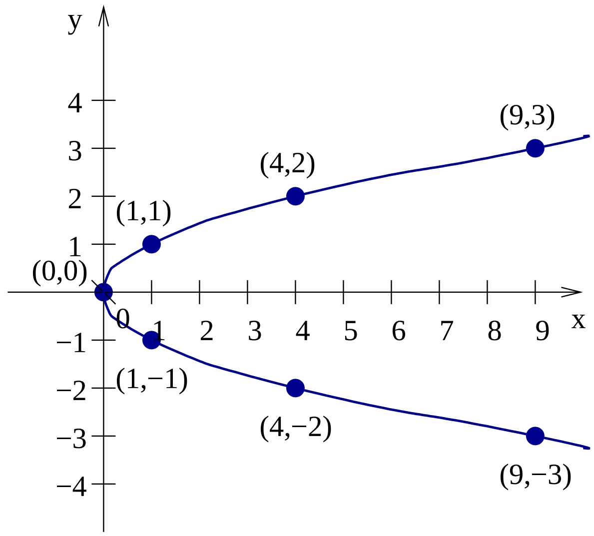
26. (5 pts each) **Extra credit problem.** Let $R=\{(x,y)\mid x,y\in\mathbb{R}, x=y^2\}$.

(a) Find the domain and the range of R.

Since any real number can be squared, y can be any real number. But x must be nonnegative, since it is the square of a real number. In fact, any nonnegative will work for x because for any $x\geq 0$, we can find a real number y such that $x=y^2$. Therefore the domain is $\mathbb{R}^{\geq 0}$ and the range is $\mathbb{R}$.

(b) Graph R.

You can find yourself a few ordered pairs in R to verify that the graph below is the correct one:



(c) Is R a function? Why or why not?

No, it is not. E.g. $(1, 1) \in R$ and $(1, -1) \in R$. Also, the graph in (b) fails the vertical line test.