GMS 91 Final Exam Solutions

May 14, 2012

See solutions to problems 1–20 in Hawkes Learning System.

21. (15 pts) Use cases to solve the absolute value inequality

$$2x + |4 - x| \leq |3x + 1|$$

where x is a real number. Express your final answer in interval notation.

We have four cases since each absolute value either leaves its argument alone, or reverses its sign.

1. $4 - x \geq 0$ and $3x + 1 \geq 0$. This happens when

$$4 \geq x$$

and

$$3x \geq -1 \iff x \geq -\frac{1}{3}.$$

If so, $|4 - x| = 4 - x$ and $|3x + 1| = 3x + 1$. So the inequality is

$$\begin{aligned} 2x + (4 - x) &\leq 3x + 1 \\ x + 4 &\leq 3x + 1 \\ 3 &\leq 2x \\ \frac{3}{2} &\leq x \end{aligned}$$

So this case yields the solution set $3/2 \leq x$ and $x \leq 4$ and $-1/3 \leq x$. Notice that whenever $3/2 \leq x$, it is also true that $-1/3 \leq x$. So we get $3/2 \leq x \leq 4$.

2. $4 - x \geq 0$ and $3x + 1 < 0$. This happens when

$$4 \geq x$$

and

$$3x < -1 \iff x < -\frac{1}{3}.$$

If so, $|4 - x| = 4 - x$ and $|3x + 1| = -(3x + 1)$. So the inequality is

$$\begin{aligned} 2x + (4 - x) &\leq -(3x + 1) \\ x + 4 &\leq -3x - 1 \\ 5 &\leq -4x \\ -\frac{5}{4} &\geq x \end{aligned}$$

So this case yields the solution set $x \leq -5/4$ and $x \leq 4$ and $x < -1/3$. Notice that whenever $x \leq -5/4$, it is also true that $x < -1/3$ and $x \leq 4$. So we get $x \leq -5/4$.

3. $4 - x < 0$ and $3x + 1 \geq 0$. This happens when

$$4 < x$$

and

$$3x \geq -1 \iff x \geq -\frac{1}{3}.$$

If so, $|4-x| = -(4-x)$ and $|3x+1| = 3x+1$. So the inequality is

$$\begin{aligned} 2x - (4-x) &\leq 3x+1 \\ 3x - 4 &\leq 3x+1 \\ -4 &\leq 1 \end{aligned}$$

Of course, $-4 \leq 1$ is always true. So the only conditions in this case are $4 < x$ and $-1/3 \leq x$. Whenever $4 < x$, it is also true that $-1/3 \leq x$. So we get $4 < x$.

4. $4 - x < 0$ and $3x + 1 < 0$. This happens when

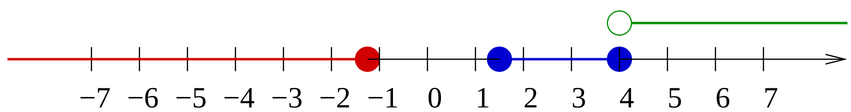
$$4 < x$$

and

$$3x < -1 \iff x < -\frac{1}{3}.$$

Notice that you cannot have both $4 < x$ and $x < -\frac{1}{3}$. So this case yields no solution.

I will combine these results on the number line:



That is $x \leq -5/4$ or $3/2 < x$. Or in interval notation $x \in (-\infty, -5/4] \cup [3/2, \infty)$.

22. (a) (3 pts) State the definition of a relation.

A relation R from a set S to a set T is a set of ordered pairs (s, t) where s is an element of S and t is an element of T.

(b) (3 pts) State the definition of a function.

A function f from a set S to a set T is a relation from S to T such that each element of S appears exactly once as the first coordinate of an ordered pair.

(c) (5 pts) Give an example of a function. Explain why your example satisfies the definition of a function.

Let S be the set of SDSU students. Let T be the set of 9-digit numbers. Let

$$f = \{(x, y) \mid x \in S, y \in T, y \text{ is } x\text{'s Red ID number}\}.$$

Then f is a function because every SDSU student has exactly one ID number.

(d) (6 pts) Give an example of a relation that is not a function. Explain why your example satisfies the definition of a relation but not that of a function.

Let S be the set of SDSU students. Let T be the set classes at SDSU. Let

$$g = \{(x, y) \mid x \in S, y \in T, x \text{ has taken } y\}.$$

Then g is a relation because it is a set of ordered pairs. But g is not a function because a typical SDSU student has taken more than one class, so they would appear more than once as the first coordinate of an ordered pair.

23. (5 pts each)

(a) Factor the polynomial $3y^3 + 16y^2 - 35y$ completely.

Notice that y is a common factor. So

$$3y^3 + 16y^2 - 35y = y(3y^2 + 16y - 35).$$

To factor $3y^2 + 16y - 35$, we can use the AC-method. We are looking for two numbers s, t such that

$$\begin{aligned} s + t &= 16 \\ st &= 3(-35) = -3 \cdot 5 \cdot 7 \end{aligned}$$

After some trying, you will find that $s = -5$ and $t = 21$ will do. So

$$\begin{aligned} 3y^2 + 16y - 35 &= \underbrace{3y^2 + 21y}_{3y(y+7)} \underbrace{-5y - 35}_{-5(y+7)} \\ &= (3y - 5)(y + 7) \end{aligned}$$

So

$$3y^3 + 16y^2 - 35y = y(3y - 5)(y + 7).$$

(b) Use the factorization from part (a) to find the solutions of the polynomial equation $3y^3 + 16y^2 - 35y = 0$.

By the Zero Factor Law,

$$0 = 3y^3 + 16y^2 - 35y = y(3y - 5)(y + 7)$$

if and only if $y = 0$ or $3y - 5 = 0$ or $y + 7 = 0$. That is the solutions are $y = 0$, $y = 5/3$, and $y = -7$.

24. (10 pts) Let $p(x)$ be a polynomial with real coefficients and c a real number. Prove that if $(x - c)$ is a factor of $p(x)$ then $p(c) = 0$.

If $(x - c)$ is a factor of $p(x)$, then we can write $p(x) = (x - c)q(x)$ for some polynomial $q(x)$. So

$$p(c) = (c - c)q(c) = 0 \cdot q(c) = 0.$$

25. (10 pts) **Extra credit problem.** The Schlafly Bottleworks brewery store has a summer promotion and offers a 20% discount on every beer in the store. You also have to pay the standard Missouri sales tax. What is more advantageous for you: if they apply the discount first and then charge the tax, or if they charge the tax first then apply the discount? What principle of algebra do you need to use to justify your argument?

It turns out, it doesn't matter. Let us say we buy something whose original price is p and that Missouri sales tax rate is t. If the discount is applied first, then the tax, we would pay

$$p(1 - 20\%)(1 + t) = p \cdot 0.8 \cdot (1 + t).$$

If the tax is applied first, then the discount, we would pay

$$p(1 + t)(1 - 20\%) = p \cdot (1 + t) \cdot 0.8.$$

By the commutative property of multiplication, these are equal.