Math 103A Practice Exercises

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Imre Tuba

True or false questions. G is a group.

1. $M_n(\mathbb{R})$, the set of $n \times n$ real matrices is a group under addition.
2. $GL_n(\mathbb{R})$, the set of $n \times n$ invertible real matrices is a group under addition.
3. $\mathbb{C}^*$ the set of nonzero complex numbers is a group under addition.
4. $\mathbb{C}^*$ the set of nonzero complex numbers is a group under multiplication.
5. If $n \in \mathbb{N}$, the integers form a group under multiplication modulo n.
6. If $n \in \mathbb{N}$, the integers relatively prime to n do not form a group under addition modulo n.
7. The nonzero vectors in $\mathbb{R}^3$ form a group under the cross product.
8. $G = \{(x, y) \mid x, y \in \mathbb{R}\}$ is a group under componentwise addition.
9. If $e, f \in G$ satisfy $eg = g = ge$ and $fg = g = gf$ for all $g \in G$ then $e = f$.
10. If $f \in G$ and $fg = g$ for all $g \in G$ (f is called a left identity) then $gf = g$ for all $g \in G$.
11. If $a, b, c \in G$ satisfy $ab = ac$, then $b = c$.
12. If $a, b, c \in G$ satisfy $ab = ca$, then $b = c$.
13. If $a, b, c \in G$ satisfy $ab = e = ca$, then $b = c$.
14. If G is a subgroup of $\mathbb{Z}$ under addition and $a, b \in G$, then $\gcd(a, b) \in G$.
15. If $(ab)^2 = a^2 b^2$ for all $a, b \in G$ then G is abelian.
16. If $n \in \mathbb{N}$ and $n > 1$ then $(ab)^n = a^n b^n$ for all $a, b \in G$ implies that G is abelian.
17. If $(ab)^{-1} = a^{-1} b^{-1}$ for all $a, b \in G$ then G is abelian.
18. If $G = \langle a, b \rangle$ then every element of G can be written in the form $a^m b^n$ for some $m, n \in \mathbb{Z}$.
19. $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} | x, y \in \mathbb{R}, \ xy \neq 0 \right\}$ is an abelian subgroup of $GL_2(\mathbb{R})$ under multiplication.
20. H as in the previous question is in the center of $GL_2(\mathbb{R})$.
21. If H is a subgroup of G and $f \in H$ satisfies $hf = h$ for all $h \in H$, then $gf = g = fg$ for all $g \in G$.
22. If G is an infinite cyclic group and $g \in G$ is of infinite order, then $G = \langle g \rangle$.
23. If $g \in G$ then the centralizer $C(g)$ of g is an abelian subgroup of G.
24. Let $n \in \mathbb{N}$ and H be the set of n-th roots of 1 in $\mathbb{C}$. Then H is a cyclic group under multiplication.
25. Let H be the set of roots (of any order) of 1 in $\mathbb{C}$. Then H is a group under multiplication.
26. If $a, n \in \mathbb{Z}$ and $n > 1$ then $a^{\phi(n)} \equiv 1 \mod n$.
27. If $|G| = n$ is even and H and K are two distinct subgroups of order 2, then G doesn't contain an element of order n.
28. The set of rational numbers $\mathbb{Q}$ forms an infinite cyclic group under addition.
29. If $a, b \in G$ are of finite order than so is ab.
30. $H = \{A \in M_n(\mathbb{R}) \mid \det A \in \mathbb{Z}, \ \det A \neq 0\}$ is subgroup of $GL_n(\mathbb{R})$.
31. If $\alpha \in S_n$ is of odd order, then $\alpha \in A_n$.

32. The set of odd permutations in S_n is a subgroup.
33. If $\alpha \in S_n$, then $|\alpha| \leq n$.
34. If $\alpha \in S_n$ is of prime order p, then α is a p-cycle.
35. There is no element of order 4 in A_5.
36. There is no element of order 4 in A_6.
37. If G is finite and $\sigma : G \to G$ is an operation preserving onto map, then σ is an automorphism.
38. If G is abelian and $\sigma(g) = g^3$, then σ is an automorphism of G.
39. Let $h \in G$ and $\sigma(g) = hgh^{-1}$. Then σ is an automorphism of G.
40. If $\sigma : G \to H$ is an isomorphism then g is in the center of G if and only if $\sigma(g)$ is in the center of H.
41. $D_3 \cong S_3$.
42. $U(p) \cong \mathbb{Z}_p$ for p a prime.
43. $U(8) \cong U(10)$.
44. $U(8) \cong U(12)$.
45. Let $\sigma : G \to G$ be a homomorphism. σ is an isomorphism if and only if $\ker \sigma = \{e\}$.
46. The group G is abelian if and only if the center of G is all of G.
47. Let H be a subgroup of G. Then $aH = bH$ if and only if $ab^{-1} \in H$.
48. The map $\sigma : G \to G$ defined by $\sigma(g) = g^2$ is a homomorphism if and only if G is abelian.
49. Let H be a subgroup of the finite group G. The number of left cosets of H in G is the same as the number of right cosets of H in G.
50. Let H be a subgroup of G. Then $aH = bH$ if and only if $Ha = Hb$.
51. If H is a subgroup of the infinite cyclic group G then either $H = \{e\}$ or H is also an infinite and cyclic.

Longer questions:

1. Find the centralizer of $(1\,2)$ in S_4.
2. List all possible disjoint cycle structures for the elements of A_5.
3. List all of the possible disjoint cycle structures for the elements of S_6. How many elements does each class contain?
4. Let G be a finite abelian group such that $g^p = e$ for all $g \in G$. Prove that if H is a subgroup of G and $g \notin H$, then $H \cap \langle g \rangle = \{e\}$.
5. Let H and K be subgroups of G. Prove that $HK = \{hk \mid h \in H,\ k \in K\}$ is a subgroup of G if and only if $HK = KH$. Conclude that HK is a subgroup if G is abelian.
6. Let H be a subgroup of G and $N = \{g \in G \mid gH = Hg\}$. N is called the normalizer of H. Prove that N is a subgroup of G and H is a normal subgroup of N.
7. Let H be a subgroup of G and $g \in G$. Prove that gHg^{-1} is a subgroup of G. What can you say about gHg^{-1} if H is normal.
8. Let H be a subgroup of G and $N = \bigcap_{g \in G} gHg^{-1}$. Prove that N is a normal subgroup of G.
9. Let H be a subgroup of G. Suppose $aH = bH$ if and only if $Ha = Hb$. Prove H is normal.
10. Let $a, b \in G$ such that $a^5 = e$ and $ab = b^2 a$. Find all possible values of $|b|$.

11. Let H and K be subgroups of a finite group G. Prove that the number of elements of $HK = \{hk \mid h \in H,\ k \in K\}$ is $\frac{|H||K|}{|H \cap K|}$.
12. Show $S_n = \langle (1\,2), (1\,3), \ldots, (1\,n) \rangle$. (Hint: you may want to do the next problem first.)
13. Show $S_n = \langle (1\,2), (2\,3), \ldots, (n-1\,n) \rangle$. (Hint: prove that the RHS contains all 2-cycles.)
14. Show $S_n = \langle (1\,2), (1\,2\,\ldots\,n) \rangle$. (Hint: use the previous problem.)
15. Show $S_n = \langle (1\,2), (2\,\ldots\,n) \rangle$. (Hint: use the previous problem.)
16. Let G be a finite group and H a subgroup. Let G act on the right cosets of H by right multiplication. If $g \in G$, what's the orbit of Hg? What's the order of the stabilizer? What's the stabilizer? (Hint: you are looking for $a \in G$ such that $Hga = Hg$, that is $Hgag^{-1} = H$.)
17. Let G be a finite group and let G act on itself by conjugation. What is the orbit of g, if g is in the center of G? Prove that for an arbitrary $g \in G$, the stabilizer is the centralizer of g. What can you say about the orbit and the stabilizer of an element g if G is abelian?
18. Let S_n act on itself by conjugation. Prove that the orbit of an element $\sigma \in S_n$ consists of those elements of S_n with the same disjoint cycle structure. (Hint: Recall how conjugation works in a permutation group.)
19. Let G be a group and S the set of subgroups of G. Let G act on S by conjugation. If N is a normal subgroup of G, what can you say about the orbit and the stabilizer of N? What can you say about a subgroup H, whose stabilizer is $\{e\}$? Note that G need not be finite in this problem.