Math 103B Final Solutions

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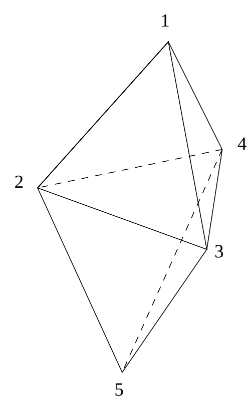
1. True. There are only two vertices with an odd number of edges, so an Euler path is not out of the question. In fact, try the path $1 \to 2 \to 3 \to 1 \to 4 \to 3 \to 5 \to 4 \to 2 \to 5$ on the figure below.



2. False. $C_{12} \cong C_4 \times C_3$ and $C_4 \not\cong C_2 \times C_2$.
3. False. Try $\mathbb{Z}_3 \times \mathbb{Z}_3$.
4. False. Consider $I = \langle 2 \rangle$ and $J = \langle 3 \rangle$ in $\mathbb{Z}$. Then $3, 2 \in I \cup J$, but $3 - 2 = 1 \notin I \cup J$.
5. True. You already know that the intersection of two subgroups is a subgroup, so all you have to verify is that if $a \in I \cap J$, then $a \in I$ and $a \in J$, so $ra \in I$ and $ra \in J$, hence $ra \in I \cap J$ for all $r \in R$; and similarly for ar.
6. False. The identity matrix is diagonal so it's in S. If S were an ideal it would have to be all of R, but not all matrices are diagonal.
7. False. $x^2+1 \in S$ and $x^2 \in S$, hence if S were an ideal, it would have to contain $1 = (x^2+1)-x^2$, which is of degree 0.
8. True. If $|G|$ had any prime divisor $p \neq 3$, then G would have an element of order p.
9. True. In this ring, $x^2 + \langle x^2+1 \rangle = -1 + \langle x^2+1 \rangle$, hence $x^4 + \langle x^2+1 \rangle = 1 + \langle x^2+1 \rangle$.
10. False. $2 + \langle x^2+1 \rangle$ doesn't have an inverse.
11. True. Every element is of the form $ax + b + \langle x^2+1 \rangle$ with $a, b \in \mathbb{Q}$, and its inverse is $(-ax + b)/(a^2+b^2) + \langle x^2+1 \rangle$ unless $a = b = 0$.
12. False. $(1,0)(0,1) = (0,0)$.
13. True. The characteristic is the order of 1 in the additive group, so it divides 27, but the characteristic of any integral domain is 0 or prime, so it must be 3 in this case.
14. $\langle x^2+1 \rangle = \{(x^2+1)\,f(x) \mid f \in \mathbb{Z}[x]\}$, so the degree of any nonzero polynomial in $\langle x^2+1 \rangle$ is at least 2. Hence $2 \notin \langle x^2+1 \rangle$, but $2 \in \langle x^2+1,\, 2 \rangle$. On the other hand, $1 \notin \langle x^2+1,\, 2 \rangle$. If 1 were in there, it would have to be $1 = (x^2+1)\,f(x) + 2\,g(x)$ for some $f, g\mathbb{Z}[x]$. Substituting $x = i$ in this last equation, we get $1 = 2\,g(i)$. This cannot happen as the real part of $2\,g(i)$ is even.

15. $400 = 2^4\, 5^2$ and 4 has 5 partitions, while 2 has only 2. So we have 10 different abelian groups of order 400. They are

$$\begin{array}{cc} \mathbb{Z}_{16} \times \mathbb{Z}_{25} & \mathbb{Z}_{16} \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} & \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{25} & \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} & \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5. \end{array}$$

16. The order of (a, b) is $\mathrm{lcm}(|a|, |b|)$. For this to be 5, one of $|a|$ and $|b|$ must be 5, while the other is 1 or 5. C_{125} and C_{25} each contain four elements of order 5 and one element of order 1. Hence we have 25 combinations of a, b with $|a|, |b| = 1, 5$, but one of them is the identity, so 24 of them are of order 5.

17. Let's use Burnside's theorem. The symmetry group is generated by R_{72° and is cyclic of order 5. The identity fixes every cup, so we could choose any of the three colors for each, which gives us 3^5 choices. The rotations $R_{72^\circ}, R_{144^\circ}, R_{216^\circ}, R_{288^\circ}$ each generate the group of rotations, so all of the cups are in the same orbit under their action. So we can only choose a color for all five of them, which gives us 3 choices. By Burnside's Theorem, there are

$$\frac{1}{5}\left(3^5 + 4 \cdot 3\right) = 51$$

ways to arrange the cups.

18. We know $\mathbb{Z}/\langle n\rangle$ is an integral domain if and only if n is prime. But a finite integral domain is a field, so $\mathbb{Z}/\langle n\rangle$ is actually a field if and only n is prime.

19. (a) i is a unit, so $\langle i\rangle = \mathbb{Z}[i]$, so it is not a proper ideal, hence neither prime nor maximal. The quotient ring only has 0 in it, so it has no units at all.

    (b) Note that $1 \notin \langle 1+i\rangle$ because $|1|^2 = 1$, whereas the typical element of the ideal has $|(1+i)\,r|^2 = |1+i|^2\,|r|^2 \geq 2$. But $2 = (1+i)(1-i) \in \langle 1+i\rangle$, so $2 + \langle 1+i\rangle = 0 + \langle 1+i\rangle$. Also $i + \langle 1+i\rangle = -1 + \langle 1+i\rangle$ so the only distinct elements of $\mathbb{Z}[i]/\langle 1+i\rangle$ are $0 + \langle 1+i\rangle$ and $1 + \langle 1+i\rangle$. That is $\mathbb{Z}[i]/\langle 1+i\rangle \cong \mathbb{Z}_2$, which is a field, hence $\langle 1+i\rangle$ is both prime and maximal. The quotient has one unit.

    (c) A typical element of $\mathbb{Z}[i]/\langle 11\rangle$ is $a + bi + \langle 11\rangle$ where we can always choose $0 \leq a,\ b \leq 10$. Note that the squares mod 11 are $0, 1, 3, 4, 5, 9$ and the only way for two of these to sum to 0 mod 11 is if they are both 0. So $a^2 + b^2 \equiv 0 \mod 11$ if and only if $a \equiv b \equiv 0 \mod 11$. $\mathbb{Z}_{11}$ is a field, so any nonzero element has an inverse, hence there exists $m \in \mathbb{Z}$ such that $m(a^2+b^2) \equiv 1 \mod 11$ unless $a \equiv b \equiv 0 \mod 11$. Therefore

    $$(a + bi + \langle 11\rangle)(ma - mbi + \langle 11\rangle) = 1 + \langle 11\rangle$$

    that is every nonzero element has an inverse in $\mathbb{Z}[i]/\langle 11\rangle$. So $\mathbb{Z}[i]/\langle 11\rangle$ is a field, and $\langle 11\rangle$ is both prime and maximal. $\mathbb{Z}[i]/\langle 11\rangle$ has 120 nonzero elements and they are all units.

    (d) Observe $17 = (1+4i)(1-4i)$, so $\langle 17\rangle = \langle 1+4i\rangle\,\langle 1-4i\rangle$. Also $\langle 4+i\rangle + \langle 4-i\rangle$ contains $1 + 4i + 1 - 4i = 2$, hence it contains $8 \cdot 2 = 16$. But it also contains $17 = (1+4i)(1-4i)$, hence it contains $17 - 16 = 1$. Thus $\langle 4+i\rangle + \langle 4-i\rangle = \mathbb{Z}[i]$. By the Chinese Remainder Theorem

    $$\mathbb{Z}[i]/\langle 17\rangle \cong \mathbb{Z}[i]/\langle 1+4i\rangle \times \mathbb{Z}[i]/\langle 1-4i\rangle$$

    Now observe $\mathbb{Z}[i]/\langle 1+4i\rangle \cong \mathbb{Z}_{17}$ and $\mathbb{Z}[i]/\langle 1-4i\rangle \cong \mathbb{Z}_{17}$, so $\mathbb{Z}[i]/\langle 17\rangle \cong \mathbb{Z}_{17} \times \mathbb{Z}_{17}$, which is clearly not an integral domain, let alone a field. So $\langle 17\rangle$ is neither prime nor maximal. The number of units in $\mathbb{Z}[i]/\langle 17\rangle$ is the square of the number of units in $\mathbb{Z}_{17}$. This is $16^2 = 256$ as $\mathbb{Z}_{17}$ is a field.