Math 103B Practice Exercises

Spring 2000, Imre Tuba

Group theory:

1. Find the symmetries (such as translation, vertical flip, horizontal flip, half turn, and glide reflection) of the following objects:
   (a) The graph of $y = \tan^2 x$.
   (b) The graph of $y = 1/\sin^2 x$.
   (c) The graph of $|y| = \cot x$.
   (d) ... 1 1 1 1 1 ...
   (e) ... 3 3 3 3 3 ...
   (f) ... 6 6 6 6 6 ...
   (g) ... 8 8 8 8 8 ...
   (h) ... ˜ ˜ ˜ ˜ ˜ ...
   (i) ... M M M M M ...
2. Find the number of different ways to color the faces of an octahedron with 2, 3, or 4 colors.
3. You are playing with one of those plastic geometry demo sets which consists of regular polygons that can be attached at the edges to build polyhedra. You want to construct a cube and you have 4 red and 4 blue squares. How many different cubes can you make? What if you had 3 red, 3 blue, and 3 green squares?
4. You want to make a necklace with 10 beads. You have 6 white and 6 black beads. How many different necklaces can you make?
5. Decide whether the following Cayley digraphs have Hamiltonian paths and Hamiltonian circuits. Justify.
   (a) $\mathrm{Cay}(\{(0,1),(1,0)\}, \mathbb{Z}_2 \oplus \mathbb{Z}_3)$
   (b) $\mathrm{Cay}(\{(0,2),(1,0)\}, \mathbb{Z}_2 \oplus \mathbb{Z}_3)$
   (c) $\mathrm{Cay}(\{(0,1),(1,0)\}, \mathbb{Z}_2 \oplus \mathbb{Z}_4)$
   (d) $\mathrm{Cay}(\{(0,1),(1,0)\}, \mathbb{Z}_4 \oplus \mathbb{Z}_6)$
   (e) $\mathrm{Cay}(\{(0,0,1),(0,1,0),(1,0,0)\}, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$
   (f) $\mathrm{Cay}(\{(1,()),(0,(1\,2)),(0,(1\,3))\}, \mathbb{Z}_2 \oplus S_3)$
   (g) $\mathrm{Cay}(\{(1,()),(0,(1\,2\,3)), \mathbb{Z}_2 \oplus A_3)$
   (h) $\mathrm{Cay}(\{(1,e),(0,R_{120^\circ}),(0,F)\}, \mathbb{Z}_2 \oplus D_3)$
   (i) $\mathrm{Cay}(\{(1,0),(0,1)\}, \mathbb{Z}_m \oplus \mathbb{Z}_n)$ when $\gcd(m,n) = 2$.
6. Let G be a group with center Z. Prove that if G/Z is cyclic, then G is abelian.
7. How many elements of order p does $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ have?
8. How many isomorphism classes of abelian groups of order 72 are there? List them.
9. Let G be a finite abelian group such that $g^p = e$ for all $g \in G$. Prove
   (a) If H is a subgroup of G and $g \notin H$, then $H \cap \langle g \rangle = \{e\}$.
   (b) If $o(G) > 1$, then $G \cong \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{k \text{ times}}$ for some k.
10. Prove that any group of order 10 is isomorphic to either $\mathbb{Z}_{10}$ or D_5. What can you say about groups of order 14? Can you prove in general that a group of order $2p$ is either cyclic or isomorphic to D_p whenever p is a prime?

Ring theory:

1. Show that the characteristic of a field is either 0 or prime.

2. Let $I_1, I_2, \ldots, \subseteq R$ be ideals in the ring R, such that $I_1 \subseteq I_2 \subseteq \ldots \subseteq R$. Show that their union is also an ideal in R. (There may be infinitely many of them.)
3. Let S be a subset of the ring R. Define the left annihilator of S to be $\operatorname{Ann}(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$. Prove that $\operatorname{Ann}(S)$ is an ideal of R if R is commutative. Prove that this is no longer true if R is noncommutative. What can you say about $\operatorname{Ann}(S)$ if R is an integral domain? Now, let I be an ideal of the (noncommutative) ring R. Prove that $\operatorname{Ann}(I)$ is an ideal of R.
4. Let R be a ring and I an ideal in R. Let $\pi : R \to R/I$ be a map defined by $\pi(r) = r+I$. Show that π is a group homomorphism and $\pi(rs) = \pi(r)\pi(s)$ for $r, s \in R$. What is the kernel of π?
5. Let R, I, and π be as in the previous problem, and let J be another ideal of R such that $I \subseteq J$. Prove that $\pi(J)$ is an ideal of R/I. Now, let K be an ideal of R/I. Prove that $\pi^{-1}(K) = \{r \in R \mid \pi(r) \in K\}$ is an ideal of R. Conclude that there is a one-to-one correspondence between the ideals of R containing I and the ideals of R/I.
6. Let R, I, and π be as in the previous problem. Prove that the correspondence of the previous exercise takes maximal ideals to maximal ideals and prime ideals to prime ideals.
7. A nilpotent element of a ring $r \in R$ is such that $r^n = 0$ for some $n \in \mathbb{N}$.
   (a) Let $r, s \in R$ be nilpotent and $rs = sr$. Show that $r + s$ is nilpotent.
   (b) Prove that $\operatorname{Nil}(R) = \{r \in R \mid r \text{ is nilpotent}\}$ is an ideal of R. Show that $R/\operatorname{Nil}(R)$ has no nonzero nilpotent elements.
8. Show that a ring R has no nonzero nilpotent elements if and only if $r^2 = 0$ implies $r = 0$ in R.
9. Let $R = 2\mathbb{Z}$, the ring of even integers. (Verify it's a ring.) Let $I = 4\mathbb{Z}$. Prove that I is a maximal ideal of R. Is R/I a field? Explain.
10. Let $R = (x^2+1)\,\mathbb{Z}[x] = \{(x^2+1)f(x) \mid f(x) \in \mathbb{Z}[x]\}$ and $I = (x-1)\,\mathbb{Z}[x]$. Note R is an ideal of $\mathbb{Z}[x]$, hence R is a ring. Prove that I is a prime ideal of R. (Hint I is a prime ideal of $\mathbb{Z}[x]$.) Prove that R/I is not an integral domain. Explain.
11. Prove that $\langle x^2+x+1 \rangle$ is a maximal ideal of $\mathbb{Q}[x]$. How many cube roots of 1 can you find in $\mathbb{Q}[x]/\langle x^2+x+1 \rangle$?
12. Let $I = \langle x^2+x \rangle$ in $\mathbb{Q}[x]$. Is I a prime ideal? Can you find the zero divisors in $\mathbb{Q}[x]/I$? Can you find an idempotent?
13. How many elements does the ring $R = \mathbb{Z}_3[x]/\langle x^2+x+1 \rangle$ have? Is it a field? Is it an integral domain? Is $\langle x^2+x+1 \rangle$ a maximal ideal of $\mathbb{Z}_3[x]$? Is it a prime ideal? Find the cube roots of 1 in R. Can you find a nilpotent element?
14. How many elements does the ring $R = \mathbb{Z}_5[x]/\langle x^2-1 \rangle$ have? Is it a field? Is it an integral domain? Is $\langle x^2-1 \rangle$ a maximal ideal? Is it a prime ideal? How many square roots of 1 can you find in R?
15. Let R be a ring and I an ideal. Let the radical of I be $\operatorname{Rad}(I) = \{r \in R \mid r^n \in I \text{ for some } n\}$. Prove that $\operatorname{Rad}(I)$ is an ideal of I. Show that a prime ideal P is equal to its radical.
16. A matrix is called upper triangular if all of its entries below the diagonal are 0. Let R be the ring of upper triangular 3×3 matrices with real coefficients. (Verify R is a ring). Is R commutative? Find some examples of zero divisors, units, idempotents, and nilpotents in R.
17. Let R be as in the previous problem. An upper triangular matrix is called strictly upper triangular if the diagonal entries are also 0. Prove that the strictly upper triangular matrices form an ideal I of R. Show that R/I is commutative.