Math 1101 Exam 1 Second Attempt Solutions

Oct 7, 2005

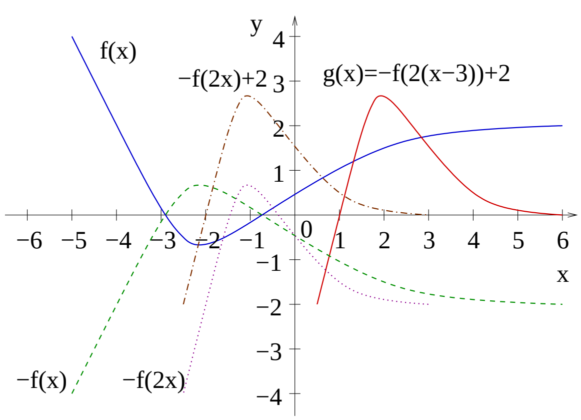
1. (8 pts)
   (a) Define what a function is.

   A function f from a set S to a set T is a rule that assigns to every element $s \in S$ an element $t \in T$.

   (b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be one-to-one.

   A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if for any $x, y \in \mathbb{R}$ such that $f(x) = f(y)$, $x = y$. Alternate definition: A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if for any $x, y \in \mathbb{R}$ such that $x \neq y$, $f(x) \neq f(y)$.

2. (7 pts) The graph of $g(x)$ given below can be obtained from the graph of $f(x)$–also given below–by a sequence of transformations, such as shifts, stretches/compressions, and reflections. If $g(x) = Cf(Ax + B) + D$, find A, B, C, D.



Notice the graph was reflected about the x-axis, shifted up by 2 units, compressed by a factor of 2 in the horizontal direction and shifted to the right by 3 units. These correspond respectively to multiplying f by -1, adding 2 to it, multiplying x by 2 and subtracting 3 from it. So $g(x) = -f(2(x-3))+2 = -f(2x-6)+2$. (Remember that multiplication is done before addition, so when you subtract 3 from x, you need to use parentheses.)

So $A = 2$, $B = -6$, $C = -1$, and $D = 2$.

3. (20 pts) Decide if the following statements are true or false and justify your answer. (Remember that you can show a statement is false by providing a counterexample, but you cannot prove a statement true by giving an example.)
   (a) If f and g are functions $\mathbb{R} \to \mathbb{R}$ then $f \circ g = g \circ f$.

   False. Let $f(x) = x^2$ and $g(x) = x + 1$. Then $f \circ g(x) = (x + 1)^2$ and $g \circ f(x) = x^2 + 1$, which are not equal.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function then f is not even.

   True. If f were even, then $f(x) = f(-x)$ for all x. In particular, $f(1) = f(-1)$. But f is one-to-one, so $1 \neq -1$ implies $f(1) \neq f(-1)$.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be an invertible function and f^{-1} its inverse. Then $f(f^{-1}(f^{-1}(f(x)))) = x$ for all $x \in \mathbb{R}$.

   True. By definition of the inverse $(f^{-1}(f(x)) = x$, and also $f(f^{-1}(x)) = x$.

(d) No even function $f : \mathbb{R} \to \mathbb{R}$ can be strictly decreasing.

True. Suppose $f(x)$ is even. Then $f(-x) = f(x)$ for all $x \in \mathbb{R}$. If $f(x)$ were strictly decreasing, then $f(-1) > f(1)$ since $-1 < 1$. But $f(x)$ is even, so we know $f(-1) = f(1)$.

4. (10 pts) Using the familiar subtraction formulas for $\sin(x-y)$ and $\cos(x-y)$, find a formula for $\cot(x-y)$ in terms of $\cot(x)$ and $\cot(y)$.

$$\begin{aligned}
\cot(x-y) &= \frac{\cos(x-y)}{\sin(x-y)} = \frac{\cos(x)\cos(y)+\sin(x)\sin(y)}{\sin(x)\cos(y)-\cos(x)\sin(y)} \\
&= \frac{\cos(x)\cos(y)+\sin(x)\sin(y)}{\sin(x)\cos(y)-\cos(x)\sin(y)} \cdot \frac{\frac{1}{\sin(x)\sin(y)}}{\frac{1}{\sin(x)\sin(y)}} \\
&= \frac{\frac{\cos(x)}{\sin(x)}\frac{\cos(y)}{\sin(y)}+1}{\frac{\cos(y)}{\sin(y)}-\frac{\cos(x)}{\sin(x)}} = \frac{\cot(x)\cot(y)+1}{\cot(y)-\cot(x)}
\end{aligned}$$

5. (15 pts) Shortly before going to jail in 1931, Al Capone puts \$10 million in a Swiss bank to shelter it from US prosecutors. In 1951, after Al Capone's death, FBI agents pick up a rumor about the money. Unfortunately, the rumor doesn't say where the money is, only that it is in an account accruing interest at a fixed rate and has grown to \$70 million in the meantime. Different banks pay interest at different rates, so one chance the federal agents have to locate the money is to figure out the interest rate.

(a) What is the annual interest rate if the bank compounds interest once a year? (It's ok to leave roots in your answer.)

Let $P(t)$ be the principal measured in \$ million at time t measured in years since 1931. Let the annual interest rate be r. Then $P(t) = P_0(1+r)^t$.

$$\begin{aligned}
70 &= 10(1+r)^{20} \\
7 &= (1+r)^{20} \\
1+r &= \sqrt[20]{7} \\
r &= \sqrt[20]{7} - 1
\end{aligned}$$

So the annual interest rate is $\sqrt[20]{7} - 1$.

(b) What is the annual interest rate if the bank compounds interest four times a year? (It's ok to leave roots in your answer.)

Let $P(t)$ be the principal measured in \$ million at time t measured in years since 1931. Let the annual interest rate be r. Then $P(t) = P_0\left(1+\frac{r}{4}\right)^{4t}$.

$$\begin{aligned}
70 &= 10\left(1+\frac{r}{4}\right)^{80} \\
7 &= \left(1+\frac{r}{4}\right)^{80} \\
1+\frac{r}{4} &= \sqrt[80]{7} \\
r &= 4(\sqrt[80]{7} - 1)
\end{aligned}$$

So the annual interest rate is $4(\sqrt[80]{7} - 1)$.

(c) Which of these two numbers do you think is bigger and why? (Don't answer this too fast, think carefully first.)

The answer to (a) should be bigger. If the annual interest rates were the same, compounding interest once a year would produce a lower return. So to produce the same return, the annual rate in (a) must be higher.

Actually, if you use a calculator, you get about 10.2% in (a) and 9.8% in (b).