Math 1101 Exam 1 Solutions

Sep 28, 2005

1. (8 pts)
   (a) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be strictly decreasing.

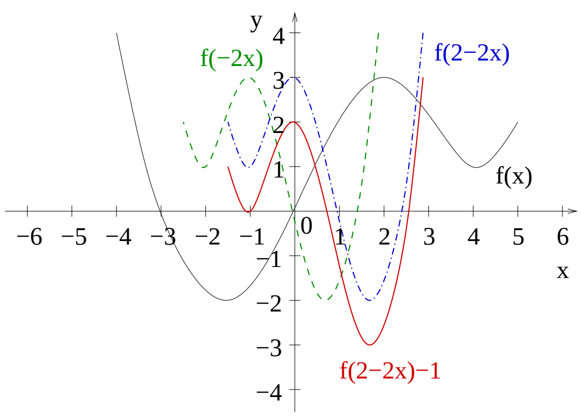
   A function $f : \mathbb{R} \to \mathbb{R}$ is strictly decreasing if for all $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$, $f(x_1) > f(x_2)$.

   (b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be even.

   A function $f\mathbb{R} \to \mathbb{R}$ is even if $f(x) = f(-x)$ for all $x \in \mathbb{R}$.

   Note: While it is true that the graph of an even function is symmetric about the y-axis, this is not the definition, but a consequence of the definition.

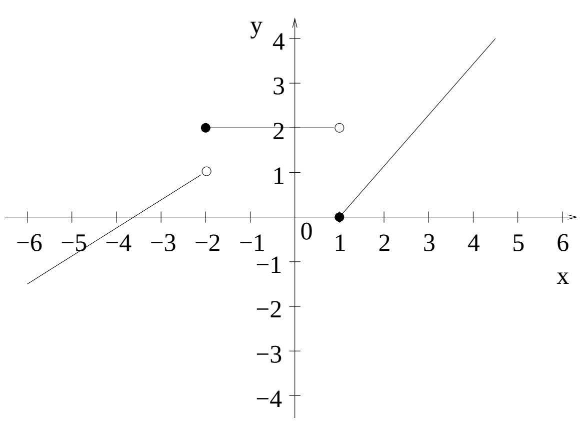
2. (6 pts) Given the graph of $f(x)$ below, sketch the graph of $f(2-2x)-1$. You may use the same set of axes or draw new ones if your prefer.



It's wise to do this one step at a time. First, notice that $f(2-2x)-1 = f(-2(x-1))-1$. This is important because the horizontal shift depends on what we do to x, not on what we do to $-2x$. We can break down the transformation into the following three steps:

1. Let $g(x) = f(-2x)$. Multiplying x reflects the graph across the y axis and compresses it in the horizontal direction by a factor of 2.
2. Let $h(x) = g(x-1) = f(2-2x)$. Subtracting 1 from x shifts the graph to the right by 1 unit.
3. $k(x) = h(x) - 1 = f(2-2x) - 1$. This shifts the graph down by 1 unit.

3. (6 pts) Below is the graph of a piecewise defined function. Write a formula for the function.



You can write down the equations of the line segments by determining the slope first as $\Delta y/\Delta x$, then substituting any point on the line segment. E.g. the for the leftmost one, y increases by 2 as x goes from -5 to -2, so the slope is $2/3$. Now substitute $(-2, 1)$ into $y = 2/3\,x + b$ to get $b = 7/3$.

$$f(x) = \begin{cases} \frac{2}{3}x + \frac{7}{3} & \text{if } x < -2 \\ 2 & \text{if } -2 \le x < 1 \\ x - 1 & \text{if } 1 \le x \end{cases}$$

4. (20 pts) Decide if the following statements are true or false and justify your answer. (Remember that you can show a statement is false by providing a counterexample, but you cannot prove a statement true by giving an example.)

   (a) If f and g are both increasing functions, so is $f + g$.

   This is true. Let $x_1 < x_2$. Then $f(x_1) \leq f(x_2)$ and $g(x_1) \leq g(x_2)$. Therefore
$$(f+g)(x_1) = f(x_1) + g(x_1) \leq f(x_2) + g(x_2) = (f+g)(x_2).$$
   Since this holds for any $x_1 < x_2$, $f + g$ is indeed increasing.

   (b) If f and g are both strictly decreasing functions, then $f \circ g$ is strictly increasing.

   This is true. Let $x_1 < x_2$. Then $g(x_1) > g(x_2)$. Let $y_1 = g(x_1)$ and $y_2 = g(x_2)$. Since $y_2 < y_1$ $f(y_2) > f(y_1)$. Therefore
$$f \circ g(x_1) = f(g(x_1)) = f(y_1) < f(y_2 = f(g(x_2)) = f \circ g(x_2).$$
   Since this holds for any $x_1 < x_2$, $f \circ g$ is indeed strictly increasing.

   (c) If f and g are both even functions, then fg is also even.

   This is true. We know that $f(-x) = f(x)$ and $g(-x) = g(x)$ for any x. Then
$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x).$$
   This is true for any x, so fg is an even function.

   (d) If f and g are both odd functions, then $f \circ g$ is also odd.

   This is true. We know that $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for any x. Then
$$f \circ g(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -f \circ g(x).$$
   This is true for any x, so $f \circ g$ is an odd function.

5. (10 pts) Using the familiar addition formulas for $\sin(x+y)$ and $\cos(x+y)$, find a formula for $\tan(x+y)$ in terms of $\tan(x)$ and $\tan(y)$.

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)} \\
&= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)} \cdot \frac{\frac{1}{\cos(x)\cos(y)}}{\frac{1}{\cos(x)\cos(y)}} \\
&= \frac{\frac{\sin(x)}{\cos(x)} + \frac{\sin(y)}{\cos(y)}}{1 - \frac{\sin(x)}{\cos(x)}\frac{\sin(y)}{\cos(y)}} = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}
\end{aligned}$$

6. (10 pts)

(a) Somewhere in a galaxy far far away, in the year 5836, the Empire undertakes to build a defensive shield for the death star it is constructing. The shield is projected from the moon Endor. Always sensitive about the environment, Darth Vader orders imperial scientists to study the impact of the electromagnetic radiation the shield emits on the local population of Ewoks. The scientists' initial survey reveals that the colony consists of 10000 Ewoks. When the scientists return in 5839, they find 11000 Ewoks. They assume that the population changes according to a continuous exponential model. Find a function which gives the number of Ewoks in the year t.

It will be convenient measure time starting in 5836, so let's replace t with $T = t - 5836$. We are told this is continuous exponential model, so it will look like

$$P(t) = P_0 e^{rT} = P_0 (e^r)^T$$

where r is the rate of growth. We know

$$\begin{aligned} 10000 &= P(0) = P_0 e^{0r} = P_0 \\ 11000 &= P(3) = P_0 e^{3r} = P_0 (e^r)^T \end{aligned}$$

We don't really need to know r, we just need to be able to write down the function, so let's substitute $a = e^r$. The first equation tells us $P_0 = 10000$. Using the second equation:

$$\begin{aligned} 11000 &= 10000 a^3 \\ \frac{11}{10} &= a^3 \\ \sqrt[3]{\frac{11}{10}} &= a \end{aligned}$$

So the model is

$$P(t) = 10000 \left(\sqrt[3]{\frac{11}{10}} \right)^{t-5836}.$$

This was a somewhat lengthy computation, so there is always a chance we might have made a mistake. Let's check the answer:

$$P(5836) = 10000 \left(\sqrt[3]{\frac{11}{10}} \right)^{0} = 10000 \qquad \checkmark$$

$$P(5839) = 10000 \left(\sqrt[3]{\frac{11}{10}} \right)^{3} = 10000 \, \frac{11}{10} = 11000 \qquad \checkmark$$

(b) Upon closer research, the scientists find that the reproductive cycle of the Ewoks is highly seasonal, and baby Ewoks are born only twice a year. How would you modify your model to reflect this new finding?

Again, let's use $T = t - 5836$ for convenience. Now we have exponential growth at some annual rate r compounded twice a year. So the model is

$$P(T) = P_0 \left(1 + \frac{r}{2} \right)^{2T}.$$

We know

$$10000 = P(0) = \left(1+\frac{r}{2}\right)^0 = P_0$$
$$11000 = P(3) = \left(1+\frac{r}{2}\right)^6$$

We don't really need to know r, we just need to be able to write down the function, so let's substitute $a = 1 + r/2$. The first equation tells us $P_0 = 10000$. Using the second equation:

$$11000 = 10000a^6$$
$$\frac{11}{10} = a^6$$
$$\sqrt[6]{\frac{11}{10}} = a$$

So the model is

$$P(T) = 10000\left(\sqrt[6]{\frac{11}{10}}\right)^{2T} = \left(\sqrt[3]{\frac{11}{10}}\right)^{T}$$
$$P(t) = 10000\left(\sqrt[3]{\frac{11}{10}}\right)^{t-5836}$$

Notice this is the same as before.

What's going on here? The answers to the two parts are the same because in both cases we really have the same model. We have an exponential function which goes through the points $(5836, 10000)$ and $(5839, 11000)$. In fact, we'd better get the same function. The only difference is how we measure the rate of growth. If we had actually computed r in the two parts, we would have different values. This is because the difference between the two parts is how the rate of growth is measured. But we were not concerned about the rate of growth, only the function which gives the population at time t, so we never needed to notice this difference.