Math 1101 Exam 2 Solutions

Nov 2, 2005

1. (10 pts)

   (a) Let $a, L \in \mathbb{R}$. Define what it means for a function f to have limit L at a.

   We say that the limit of the function f at a is L if $f(x)$ can be made arbitrarily close to L by requiring that x be sufficiently close to a.

   Note: It's essential to say that $f(x)$ gets arbitrarily close to L. Saying only that $f(x)$ gets closer and closer to L as x approaches a is not enough. For example, consider $f(x) = e^{-1/x^2}$ as $x \to 0$. As x approaches 0, $f(x)$ get closer and closer to 0. But $f(x)$ also gets closer and closer to -1, or any other negative number for that matter. But it does not get arbitrarily close to -1. $f(x)$ does get arbitrarily close to 0 as x approaches 0, which is why $\lim_x \to 0e^{-1/x^2} = 0$.

   (b) State the Squeeze Theorem. Be particularly careful to include all conditions.

   Let f, g, h be functions and $a \in \mathbb{R}$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

   and

$$f(x) \le g(x) \le h(x)$$

   for all x in some open interval surrounding a, then

$$\lim_{x\to a} g(x) = L.$$

2. (10 pts) Use induction to prove

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$$

for all $n \ge 2$.

Base case $n = 2$:

$$\frac{1}{1\cdot 2} = \frac{1}{1\cdot 2} \qquad \checkmark$$

Inductive hypothesis:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$$

for some $n \in \mathbb{N}$.

We want to prove

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Notice that using the inductive hypothesis

$$\underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n}}_{\frac{n-1}{n}} + \frac{1}{n(n+1)} = \frac{n-1}{n} + \frac{1}{n(n+1)} =$$

$$\frac{(n-1)(n+1)+1}{n(n+1)} = \frac{n^2-1+1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1}$$

3. (5 pts each) Decide if the following statements are true or false and justify your answer. (Remember that you can show a statement is false by providing a counterexample, but you cannot prove a statement true by giving an example.)
   (a) $\arcsin(\sin(x)) = x$ for all $x \in \mathbb{R}$.

   False. Counterexample: $\arcsin(\sin(\pi)) = \arcsin(0) = 0 \neq \pi$.

   Note: We noted in class that $\sin(x)$ is not a one-to-one function, therefore it has no inverse. So $\arcsin(x)$ is not the inverse of $\sin(x)$.
   The definition of the arcsin function is that for $x \in [-1, 1]$, $\arcsin(x)$ is such an angle $y \in [-\pi/2, \pi/2]$ that $\sin(y) = x$. If you define $\overline{\sin}$ to be the restriction of sin to the domain $[-\pi/2, \pi/2]$, then arcsin is the inverse of $\overline{\sin}$.

   (b) $\sin(\arcsin(x)) = x$ for all $x \in [-1, 1]$.

   True, by definition of the arcsin function, $\arcsin(x)$ is an angle $y \in [-\pi/2, \pi/2]$ such that $\sin(y) = x$.

   (c) $\log_x(y)\log_y(x) = 1$ for all $x, y > 0$.

   True.
$$\log_x(y)\log_y(x) = \frac{\ln(y)}{\ln(x)}\frac{\ln(x)}{\ln(y)} = 1$$

   (d) Let $a \in \mathbb{R}$. If $f(x) \leq g(x)$ for all $x \in \mathbb{R}$ and $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} g(x)$ also exists and
$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

   False. Let $f(x) = 0$ and $g(x) = 1/x^2$. Then $f(x) \leq g(x)$ for all $x \in \mathbb{R}$ and $\lim_{x\to 0} f(x) = 0$, but $\lim_{x\to 0} g(x)$ does not exist (it's ∞).

   Don't confuse this with the Sandwich Theorem. Here g is not sandwiched between two functions whose limits exist and are equal at a.

   (e) Let f be a function and $a \in \mathbb{R}$. Then $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist.

   False. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are unequal, $\lim_{x\to a} f(x)$ does not exist. Here is a counterexample. Let $f(x) = \operatorname{sgn}(x)$ and $a = 0$. Then $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$, but $\lim_{x\to 0} f(x)$ doesn't exist because $f(x)$ cannot be made close to <u>one</u> number L by forcing x close to 0.

4. (5 pts each) Find the following limits. Some of these may be $\pm\infty$ or may not exist at all. Be sure to carefully justify your answers.
   (a) $\displaystyle\lim_{x\to-\infty} \frac{3x^4 - 8x^3 - 2x^2 + 5}{-3x^2 - 2x + 1}$

   As x goes to $-\infty$, all lower terms of a polynomial become negligible compared to the leading term. Hence

$$\lim_{x\to-\infty} \frac{3x^4 - 8x^3 - 2x^2 + 5}{-3x^2 - 2x + 1} = \lim_{x\to-\infty} \frac{3x^4}{-3x^2} = \lim_{x\to-\infty} \frac{3x^2}{-3} = \lim_{x\to-\infty} -x^2 = -\infty$$

Alternate solution:

$$\lim_{x\to-\infty}\frac{3x^4-8x^3-2x^2+5}{-3x^2-2x+1}=\lim_{x\to-\infty}\frac{\frac{3x^4-8x^3-2x^2+5}{x^2}}{\frac{-3x^2-2x+1}{x^2}}=\lim_{x\to-\infty}\frac{3x^2-8x-2+\frac{5}{x^2}}{-3-\frac{2}{x}+\frac{1}{x^2}}$$

Since

$$\lim_{x\to-\infty}\frac{5}{x^2}=\lim_{x\to-\infty}\frac{2}{x}=\lim_{x\to-\infty}\frac{1}{x^2}=0$$

and as x goes to $-\infty$, $8x^3$, $2x^2$, and 5 become negligible compared to $3x^2$

$$\lim_{x\to-\infty}\frac{3x^4-8x^3-2x^2+5}{-3x^2-2x+1}=\lim_{x\to-\infty}\frac{3x^2}{-3}=\lim_{x\to-\infty}-x^2=-\infty$$

(b) $\displaystyle\lim_{x\to5}\frac{\sqrt{x^2-10}-x}{2x}$

Let's try to use the Limit Laws.

$$\lim_{x\to5}\frac{\sqrt{x^2-10}-x}{2x}=\frac{\lim_{x\to5}\sqrt{x^2-10}-x}{\lim_{x\to5}2x}=\frac{\sqrt{\lim_{x\to5}(x^2-10)}-\lim_{x\to5}x}{\lim_{x\to5}2x}=$$
$$\frac{\sqrt{\lim_{x\to5}x^2-\lim_{x\to5}10}-\lim_{x\to5}x}{\lim_{x\to5}2x}=\frac{\sqrt{15}-5}{10}$$

(c) $\displaystyle\lim_{x\to0^+}x\sin\left(\frac{1}{x}\right)$

Notice that $-1\le\sin(1/x)\le1$ for all $x\neq0$. Assume $x>0$ and multiply this inequality by x. (Since $x>0$, the sign doesn't flip.)

$$-x\le x\sin\left(\frac{1}{x}\right)\le x$$

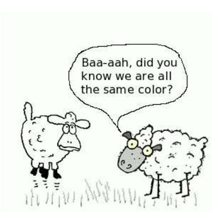
for all $x\in(0,\infty)$. Also

$$\lim_{x\to0^+}-x=0=\lim_{x\to0^+}x$$

We can now use the one-sided limit version of the Sandwich Theorem to conclude

$$\lim_{x\to0^+}x\sin\left(\frac{1}{x}\right)=0$$

5. (10 pts) **Extra credit problem.** Don't attempt this problem until you are done with everything else.



Find the mistake in the following argument and argue why it's a mistake.

The following is a proof that there exist no black sheep. First, we will prove that in any group of sheep, every sheep has the same color by doing induction on the number of sheep in the group.

It is obvious that in any group of sheep which consists of exactly one sheep, every sheep has the same color. This establishes the base case.

The inductive hypothesis is that in every group of n sheep, every sheep has the same color. Now look at a group of $n+1$ sheep. Let's pick one, set it aside, and look at the rest of the animals. They form a group of n sheep, therefore they all have the same color by the inductive hypothesis. Now we will prove that the sheep we set aside has the same color too. Let's pick another sheep and switch it with the sheep we set aside. We still have a group of n sheep, therefore they all have the same color by the inductive hypothesis. Hence the sheep we first set aside has the same color as all the others.

The above argument shows that every sheep on earth has the same color. I suppose that you've seen a white sheep before. Now you know that every other sheep must also be white. Hence there exist no black sheep despite any rumor you might have heard to the contrary.

I'll let you think more about this one. The mistake is not syntactic, such as a a group has to have more than one individual in it. (There is no reason why you couldn't consider just one sheep at a time.) It is not philosophical either, such as math is not applicable to real life. (Any scientist will assure you that it is very applicable.) It is not that experimental observations cannot be used in a proof either. (Otherwise science would not exist.)

There is really an objective mathematical mistake in the argument. If you really have no idea, try following the induction as $n = 1, 2, 3, \ldots$.