Math 1101 Exam 3 Solutions

Nov 30, 2005

1. (15 pts)

   (a) Let $f(x)$ be differentiable on the interval $[-1,1]$. Suppose $f(-1)=2$, $f(1)=-5$. Can $1/f(x)$ be a differentiable function on $[-1,1]$? If so, prove it, if not, give a counterexample.

   Since f is differentiable, it is also continuous. We know $f(-1)>0$ while $f(1)<0$. By the Intermediate Value Theorem, f must then pass through the x-axis somewhere between -1 and 1, that is there exists $-1 \le c \le 1$ such that $f(c)=0$. But then $1/f(x)$ cannot be continuous on $[-1,1]$ and therefore cannot be differentiable.

   (b) Find $\lim_{x\to\infty} x\sin(1/x)$. (Hint: multiplying by x is the same as dividing by $1/x$.)

   $$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{\frac{1}{x}}$$

   Let's substitute $h=1/x$. As $x\to\infty$, $h\to 0$. So

   $$\lim_{x\to\infty} \frac{\sin(1/x)}{\frac{1}{x}} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

   (c) Let $f:\mathbb{R}\to\mathbb{R}$ be an odd function and $a\in\mathbb{R}$. Is it true that the secant line to f between $-a$ and a always passes through the origin? If so, prove it, if not, give a counterexample.

   The equation of the secant line between $-a$ and a would be

   $$\begin{aligned} y &= \frac{f(a)-f(-a)}{a-(-a)}(x-a)+f(a) = \frac{f(a)+f(a)}{2a}(x-a)+f(a) \\ &= \frac{f(a)}{a}(x-a)+f(a) = \frac{f(a)}{a}x \end{aligned}$$

   where we used that $f(-a)=-f(a)$. When $x=0$, $y=0$, so this indeed passes through the origin.

2. (7 pts each) Find the following derivatives.

   (a) Using the quotient rule and the chain rule,

   $$\begin{aligned} \frac{d}{dx}\frac{\arcsin\left(\frac{2}{x^2}\right)}{6^{3x}} &= \frac{d}{dx}\arcsin(2x^{-2})\,6^{-3x} \\ &= \frac{1}{\sqrt{1-(2x^{-2})^2}}(-4x^{-3})6^{-3x} + \arcsin(2x^{-2})6^{-3x}\ln(6)(-3) \\ &= 6^{-3x}\left(\frac{-4x^{-3}}{\sqrt{\frac{x^4-4}{x^4}}} - 3\ln(6)\,\arcsin(2x^{-2})\right) \\ &= 6^{-3x}\left(\frac{-4x^{-3}\sqrt{x^4}}{\sqrt{x^4-4}} - 3\ln(6)\arcsin(2x^{-2})\right) \\ &= 6^{-3x}\left(\frac{-4}{x\sqrt{x^4-4}} - 3\ln(6)\arcsin\left(\frac{2}{x^2}\right)\right) \end{aligned}$$

(b) Notice that we are differentiating with respect to t, so x^t is an exponential function and $x^4 - 3x^2$ is a scalar factor.

$$\frac{d}{dt}(x^4 - 3x^2)\,x^t = (x^4 - 3x^2)\,x^t \ln(x)$$

(c)

$$\begin{aligned}\frac{d}{dx}\ln\left(\sqrt[3]{\frac{5x^3 - 3x^2 + 7}{(1-\sqrt{x})^2}}\right) &= \frac{d}{dx}\frac{1}{3}\left(\ln(5x^3 - 3x^2 + 7) - 2\ln(1-\sqrt{x})\right) \\ &= \frac{1}{3}\left(\frac{15x^2 - 6x}{5x^3 - 3x^2 + 7} - 2\frac{-\frac{1}{2\sqrt{x}}}{1-\sqrt{x}}\right) \\ &= \frac{1}{3}\left(\frac{15x^2 - 6x}{5x^3 - 3x^2 + 7} + \frac{1}{\sqrt{x} - x}\right)\end{aligned}$$

(d)

$$\begin{aligned}\frac{d}{dx}(\ln x)^{1/x} &= \frac{d}{dx}\left(e^{\ln(\ln x)}\right)^{1/x} = \frac{d}{dx}e^{\frac{1}{x}\ln(\ln x)} \\ &= e^{\frac{1}{x}\ln(\ln x)}\left(\frac{-1}{x^2}\ln(\ln x) + \frac{1}{x}\frac{1}{\ln x}\frac{1}{x}\right) = \frac{(\ln x)^{1/x}}{x^2}\left(\frac{1}{\ln x} - \ln(\ln x)\right)\end{aligned}$$

3. (10 pts)

(a) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at the point $x_0 \in \mathbb{R}$.

f is differentiable at x_0 if the derivative

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

(b) Let $f(x) = 1/x$. Use the definition of the derivative to find $f'(x)$.

$$\lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0}\frac{x - (x+h)}{hx(x+h)} = \lim_{h\to 0}\frac{-h}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

4. (7 pts) Find the equation of the tangent line to the curve

$$(y^2 - 2y)(x-2)(2x-3) = 3x^3$$

at the point $(1,3)$. (Hint: The computation is easier if you substitute the point into the equation after differentiating and before solving for $\frac{dy}{dx}$.)

Using the product rule and the chain rule,

$$\begin{aligned}(2yy' - 2y')(x-2)(2x-3) + (y^2 - 2y)(2x-3) + 2(y^2 - 2y)(x-2) &= 9x^2 \\ (6y' - 2y')(-1)(-1) + (9-6)(-1) + 2(9-6)(-1) &= 9 \\ 4y' - 3 - 6 &= 9 \\ y' = \frac{18}{4} &= \frac{9}{2}\end{aligned}$$

The equation of the tangent line is

$$y - 3 = \frac{9}{2}(x - 1)$$
$$y = \frac{9}{2}x - \frac{3}{2}$$

(As a quick reality check, you can verify that the point $(1, 3)$ is indeed on the line.)

5. (15 pts) **Extra credit problem.** Don't attempt this problem until you are done with everything else.

   A fixed point of the function f is some element c in the domain of f for which $f(c) = c$.

   (a) Use the Intermediate Value Theorem to show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (Hint: consider the function $g(x) = f(x) - x$.)

   I don't want to rob you of the joy of discovery, so I'll let you think more about this. Fixed points are quite an interesting topic. For example, they appear in chaos theory. A generalization of this statement to functions on the surface of a sphere is sometimes called the Hairy Ball Theorem. It has important real life applications, such as it implies that at any time, there is a point on the surface of the earth where the wind is calm (meteorology) and you cannot comb a hedgehog without leaving some of its spikes pointing up punk style (zoology, hairdressing science).

   (b) Graph a function $f : [0, 1] \to [0, 1]$ which has no fixed point. (Such a function must be discontinuous by the previous part.)

   This is quite easy to do. You should be able to come up with a graph after some experimentation. You may try drawing a graph of a continuous function $f : [0, 1] \to [0, 1]$ which has no fixed point. You can't, but as you try, you'll discover why not.

   (c) Must a continuous function $f : \mathbb{R} \to \mathbb{R}$ have a fixed point? If you think yes, give a proof, if not, give a counterexample.

   The answer is no. Think about what line the graph of such a function is not allowed to cross. Once you figure this out, it should be easy to find a continuous function that doesn't cross that line.