Math 1101 Quiz Problems 2

Prepare by 11/16/05

1. Problems 1-3, 5, 8, 9, 11-14 on p. 181
2. Prove by induction that for any $n \in \mathbb{Z}^+$,
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$
3. Prove by induction that for any $n \in \mathbb{Z}^+$,
$$2 + 6 + 10 + \cdots + (4n-2) = 2n^2$$
4. Prove by induction that for any $n \in \mathbb{Z}^+$,
$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$
5. Prove by induction that $n^2 > 2n+1$ for $n \geq 3$.
6. Define the Fibonacci sequence as usual ($F_1 = F_2 = 1$ and $F_{n+1} = F_{n-1} + F_n$ for $n \geq 2$ and use induction to prove
   (a) F_n is positive for $n \in \mathbb{Z}^+$. Conclude $F_n \leq 2F_{n-1}$.
   (b) $F_{n+2} - 1 = F_1 + F_2 + \cdots + F_n$ for $n \in \mathbb{N}$.
   (c) $F_{2n+1} - 1 = F_2 + F_4 + \cdots + F_{2n}$ for $n \in \mathbb{N}$.
   (d) $F_n \leq \left(\frac{7}{4}\right)^n$
   (e) $F_n \geq \left(\frac{5}{4}\right)^n$
7. Find the mistake in the following argument.

   We will prove that any two positive integers are equal. First, we will prove that if the maximum of two positive integers is any positive integer n, then they are equal. The proof proceeds by induction on n. The base case is $n = 1$. If $a, b \in \mathbb{Z}+$ are such that their maximum is 1, then they both must be equal to 1, so indeed $a = b$. The inductive hypothesis is that if $a, b \in \mathbb{Z}^+$ and their maximum is n, then $a = b$. Now let $a, b \in \mathbb{Z}^+$ such that their maximum is $n+1$. Then the maximum of $a-1$ and $b-1$ is n, so by the inductive hypothesis, $a - 1 = b - 1$. Hence $a = b$.

   Now given any two positive integers, their maximum is some positive integer n, so by the previous result, they must be equal.
8. In the following cases, compute the first few functions and guess an expression for $f_n(x)$. Use induction to prove that your formula is correct.
   (a)
$$f_0(x) = \frac{x}{x+2}$$
$$f_{n+1}(x) = f_0(f_n(x)) \qquad n \in \mathbb{N}$$
   (b)
$$f_0(x) = \frac{x}{x-1}$$
$$f_{n+1}(x) = f_0(f_n(x)) \qquad n \in \mathbb{N}$$
   (c)
$$f_0(x) = x$$
$$f_{n+1}(x) = \frac{f_n(x)}{f_n(x)+1} \qquad n \in \mathbb{N}$$