Math 110, Homework Solution Set 1

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The following solutions are courtesy of Greg Leibon and Dell Kronewitter. I only compiled them together.

p 13. 1: Here we are looking for he steady state temperature of a slab bounded between $x = 0$ and $x = c$ with faces at temperatures $u = 0$ ands $u = u_0$.

The solution to this problem varies only in the x direction and as such reduces to one one the first differential equation any one ever plays with, namely:

$$\frac{d^2u}{dx^2} = 0.$$

After integrating once we obtain $\frac{du}{dx} = b$ and integrating again we get $u(x) = bx + d$ for some constants b and d. Our goal is to solve this using the boundary conditions $u(0) = c0 + d = 0$ (giving $d = 0$) and $u(c) = bc + 0 = u_0$ (giving $b = \frac{u_0}{c}$).

So plugging in, we find the solution is $u(x) = \frac{u_0}{c}x$.

Note now by Fourier's law ((1) on page 4) $\Phi_0 = -K\frac{-\partial u}{\partial x} = K\frac{u_0}{c}$.

p. 14. 4: This problem is very similar to the first problem except with concentric spheres (of radius a and b with $a < b$) bounding the region. The boundary surfaces are maintained at the constant temperatures $r(a) = 0$ and $r(b) = u_0$. So the equation describing the heat should depend only on the radius and be independent of the spherical variables ϕ and θ. Using (22) on page 13 and the fact our equation is also independent of θ we have:

$$\Delta = r\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial u}{\partial\theta}\right) = \frac{\partial^2}{\partial r^2}(ru) = 0$$

Is the steady state equation.

From this we have by integrating (as in problem one) $ru(r) = cr + d$ or $u(r) = \frac{cr+d}{r} = c + \frac{d}{r}$. From the boundary conditions $u(a) = \frac{ca+d}{a} = 0$ (giving $c = -\frac{d}{a}$) ; and $u(b) = \frac{-\frac{d}{a}b+d}{b} = u_0$ (giving $d = \frac{abu_0}{a-b}$). So

$$u(r) = \frac{-bu_0}{a-b} + \frac{abu_0}{r(a-b)} = \frac{b}{a-b}(-u_0 + \frac{a}{r}).$$

p. 14. 5: Now we we will change a boundary conditions in the previous problem ; namely, replace the boundary condition at the surface $r = b$ with its Newton's law analog of $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = h(T - u)$.

Using the notation of the previous solution, we have $\frac{\partial u}{\partial r}(b) = \frac{-d}{r^2}(b) = \frac{-d}{b^2} = h(T - (c + \frac{d}{b}))$ (giving $d = \frac{-hb^2aT}{a-abh+b^2h}$). So

$$u(r) = \frac{-d}{a} + \frac{d}{r} = \frac{hb^2T}{a-abh+b^2h} + \frac{-hb^2aT}{a-abh+b^2h}\frac{1}{r}.$$

$$= \frac{hb^2T}{a-abh+b^2h}(1 - \frac{a}{r})$$

p. 16. 13: In this problem we have the upper-hemisphere of radius 1 with an insulated bottom (hence the boundary condition $\frac{\partial u}{\partial z} = 0$ when $z = 0$).

We want to show $\frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) = 0$

(which is clear) via the formula

$$\frac{\partial u}{\partial \theta} = -\rho \frac{\partial u}{\partial z} + z \frac{\partial u}{\partial \rho}.$$

Which indeed immediately gives $\frac{\partial u}{\partial \theta} = -\rho frac \partial u \partial z + z \frac{\partial u}{\partial \rho} = -\rho 0 + 0 \frac{\partial u}{\partial \rho} = 0$ (since when $\theta = \frac{\pi}{2}$ we have $z = 0$, which from the boundary conditions also gives that $\frac{\partial u}{\partial z} = 0$).

To show this formula use the chain rule

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} + \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial \theta}$$

and from (16) on page 12: $\frac{\partial z}{\partial \theta} = \frac{\partial (r\cos(\theta))}{\partial \theta} = -r\sin(\theta) = -\rho$ and $\frac{\partial \rho}{\partial \theta} = \frac{\partial (r\sin(\theta))}{\partial \theta} = r\cos(\theta) = z$.

Together we indeed get the formula $\frac{\partial u}{\partial \theta} = -\rho \frac{\partial u}{\partial z} + z \frac{\partial u}{\partial \rho}$.

p. 22. 1: Well the non-homogeneous wave equation is $y_{tt} = a^2 y_{xx} - g$ ((7) on p.19) and if it's static (i.e. just resting) then $y_t = 0$ giving $y_{tt} = 0$ as well. So we are reduced to the equation $a^2 y_{xx} - g =$ or $y_{xx} = \frac{g}{a^2}$. The fixing of end points to the x-axis means that the boundary conditions are $y(0) = 0$ and $y(2c) = 0$.

To solve this problem integrate twice to get

$$y(x) = \frac{g}{2a^2}x^2 + bx + d$$

and plug in the end points giving $0 = y(0) = d$ and $0 = y(2c) = \frac{g}{2a^2}4c^2 + 2cb$ or $b = \frac{-gc}{a^2}$. So the solution is

$$y(x) = \frac{g}{2a^2}x^2 - \frac{gc}{a^2}x = \frac{g}{2a^2}x(x - 2c)$$

So $y(x)$ is the needed parabola with minimum at c of depth $\frac{-c^2 g}{2a^2}$.

p. 22. 3: To do this problem note that on page 12 (10) we are given the planar Δ in polar coordinates as $\Delta z = \frac{1}{\rho}(\rho z)_\rho + \frac{1}{\rho^2} z_{\theta\theta}$. (Really we are given the cylindrical Laplacian which when restricted to the plane is this).So the static wave equation for a membrane becomes

$$0 = a^2 \frac{1}{\rho}(\rho z_\rho)_\rho + \frac{1}{\rho^2} z_{\theta\theta}.$$

When there is nothing going on in the θ direction (as in the problem) the equation reduces to

$$0 = a^2 \frac{1}{\rho}(\rho z_\rho)_\rho.$$

Noting the radius is never zero (and $a \neq 0$)we find this is equivalent to solving

$$0 = (\rho z_\rho)_\rho$$

with our boundary conditions are given as $z(1) = 0$ and $z(\rho_0) = z_0$.

Integrating we get $\rho z_{rho} = b$; and integrating again we get$z(\rho) = b\ln(\rho) + c$.

So from the first initial condition $0 = z(1) = b \cdot 0 + c$ giving $c = 0$. The second condition gives $z_0 = z(\rho_0) = bln(\rho_0)$ or $b = \frac{z_0}{\ln(\rho_0)}$. So one gets

$$z(\rho) = \frac{z_0 \ln(\rho)}{\ln(\rho_0)}$$

as needed.

p. 32. 1: (a) Integrating twice and noting "constants" can depend on y gives

$$u(x,y) = x^3y + xh(y) + g(y).$$

So the first boundary condition gives $u(0,y) = g(y) = y$ and $u_x(1,y) = 3y + h(y) = 0$ gives $h(y) = -3y$. So the solution must be $u(x,y) = x^3y - 3yxh + y$.

(b) $u_{xy}(x,y) = 2x$ gives $u(x,y) = x^2y + g(y) + h(x)$. So $u(0,y) = g(y) + h(0) = 0$ gives $g(y) = -h(0) = -c$ and $u(x,0) = h(x) - c = x^2$ gives $h(x) = x^2 + c$. So $u(x,y) = x^2y - c + x^2 + c = x^2y + x^2$ as needed.

p. 32. 2.(b): $u_{xx} + 2x^2u_{xy} + yu_{yy} = 0$ has discriminant $(2x^2)^2 - 4(y)(1) = 4x^4 - 4y$ and so is parabolic on $y = x^4$ (discriminant $= 0$), elliptic above the curve (discriminant < 0), and hyperbolic below the curve (discriminant > 0).

p. 33. 4: From p.28 (11) we know the general solution to the one dimensional wave equation on the whole line is $y(x,t) = \phi(x+at) + \psi(x-at)$ for some twice differentiable ϕ and ψ. Let ϕ_u and ψ_u denote the derivatives of the functions ϕ and ψ respectively. Using the initial condition $y(0,x) = 0$ we get $\phi(x) = -\psi(x)$. Similarly using $y_t(0,x) = 0$ (and the chain rule $\frac{d\phi(x+at)}{dt} = a\phi_u(x+at)$ and $\frac{d\psi(x-at)}{dt} = -a\psi_u(x+at)$) we find

$$y_t(0,x) = a\phi_u(x) - a\psi_u(x) = 2a\phi_u(x) = g(x).$$

(Note for the second derivative of ϕ to exist we must allow g to be differentiable at this point – which was not mentioned.) Now observe by the fundamental theorem of calculus that

$$\phi(x+at) - \phi(x-at) = \int_{x-at}^{x+at} \phi_u du.$$

Using the above observation that $\phi_u(x) = \frac{g(x)}{2a}$ we achieve

$$\phi(x+at) - \phi(x-at) = \frac{1}{2a}\int_{x-at}^{x+at} g(x)dx.$$

Now recalling $\phi(x-at) = -\psi(x-at)$ and $y(t,x) = \phi(x+at) + \psi(x-at)$, we find

$$y(t,x) = \phi(x+at) + \psi(x-at) = \frac{1}{2a}\int_{x-at}^{x+at} g(x)dx$$

as needed.

p. 32. 6: Take $u = e^{(\lambda x + \mu y)}$ and note $u_{xx} = \lambda^2 u$; $u_{xy} = \lambda\mu u$; $u_{xy} = \mu^2 u$; $u_x = \lambda u$; $u_y = \mu u$.

Now substitute $u = e^{(\lambda x + \mu y)}$ into the equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$ and to get $u(A\lambda^2 + B\lambda\mu + C\mu^2 + D\lambda + E\mu + F)$. So this choice of $u(x,y)$ satisfies the equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + FU = 0$$

if and only if λ and μ satisfy the algebraic relationship

$$A\lambda^2 + B\lambda\mu + C\mu^2 + D\lambda + E\mu + F = 0.$$

p. 34. 7: This problem requires only the remembering of some simple properties of the deri vative. Namely the second derivative is linear or $\frac{\partial^2}{\partial x^2}\sum_{n=1}^N c_n u_n = \sum_{n=1}^N c_n \frac{\partial^2 u_n}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}\sum_{n=1}^N c_n u_n = \sum_{n=1}^N c_n \frac{\partial^2 u_n}{\partial y^2}$.

From this observe

$$\begin{aligned} \Delta u &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sum_{n=1}^N c_n u_n = \left(\frac{\partial^2}{\partial x^2}\right)\sum_{n=1}^N c_n u_n + \left(\frac{\partial^2}{\partial y^2}\right)\sum_{n=1}^N c_n u_n \\ &= \sum_{n=1}^N c_n \frac{\partial^2 u_n}{\partial x^2} + \sum_{n=1}^N c_n \frac{\partial^2 u_n}{\partial y^2} \\ &= \sum_{n=1}^N c_n \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2}\right) = \sum_{n=1}^N c_n \Delta(u_n). \end{aligned}$$

Now let's use the fact that each u_n satisfies Laplace's equation that $\Delta(u_n) = 0$, giving

$$\Delta u = \sum_{n=1}^N c_n \Delta(u_n) = \sum_{n=1}^N c_n 0 = 0;$$

which is exactly the needed fact that u satisfies Laplace's equation.

p. 45. 1: (a) First note that for $k = 0$ that $\int_0^\pi \cos(kx)dx = \int_0^\pi \cos(0x)dx = \pi$. Now if $k \neq 0$

$$\int_0^\pi \cos(kx)dx = \frac{1}{k}\sin(kx)|_0^\pi = \sin(k\pi) - 0 = 0$$

So from the identity in the book we have

$$\int_0^\pi \cos(nx)\cos(mx)dx = \frac{1}{2}\int_0^\pi \cos((n-m)x) - \cos((n+m)x)dx$$

So using the first computations

$$\int_0^\pi \cos(nx)\cos(mx)dx = \left\{ \begin{array}{ll} \frac{\pi}{2} & n = m \\ 0 & n \neq m \end{array} \right\}.$$

(b) Here we are to show $\{\sqrt{\frac{2}{\pi}}\} \cup \{\frac{1}{\sqrt{\pi}}\}$ is an orthonormal family. From the above note that if n and m are not ze ro then we have

$$\int_0^\pi \phi_n(x)\phi_m(x)dx = \int_0^\pi \sqrt{\frac{2}{\pi}}\cos(nx)\sqrt{\frac{2}{\pi}}\cos(mx)dx = \left\{ \begin{array}{ll} \frac{2}{\pi}\frac{\pi}{2} = 1 & n = m0 \\ \frac{2}{\pi}0 = 0 & n \neq m \end{array} \right\}.$$

To finish the problem it is necessary to see how things look when inner producted with $\phi_0 = \sqrt{\frac{1}{\pi}}$ For $n \neq 0$ (from the first computation) we have $(\phi_0, \phi_n) =$

$\int_0^\pi \sqrt{\frac{1}{\pi}}\sqrt{\frac{2}{\pi}}\cos(nx)dx = 0$ as needed. Also $(\phi_0, \phi_0) = \int_0^\pi \sqrt{\frac{1}{\pi}}\sqrt{\frac{1}{\pi}}dx = \frac{1}{\pi}\int_0^\pi dx = 1$ as need; so the entire set is an orthonormal family.

p. 46. 4: ψ_1 and ψ_2 are orthogonal on $[-1, 1]$ since

$$\int_{-1}^1 \psi_1\psi_2 xd = \int_{-1}^1 xdx = \frac{1}{2}x^2|_{-1}^1 = 1 - 1 = 0$$

For $(Bx^2 + Ax + 1, \psi_1) = 0$ implies

$$\int_{-1}^1 Bx^2 + Ax + 1dx = 0$$

or $\frac{2B}{3} + 2 = 0$ so $b = -3$. And for $(Bx^2 + Ax + 1, \psi_1) = 0$ implies

$$\int_{-1}^1 Bx^3 + Ax^2 + xdx = 0$$

or $\frac{A}{3} = 0$. So $A = 0$. So we're done.

Note: The next three problems are just computation so to make them more interesting I thought we'd take a look at some relate computations form which to derive them. First off recall our inner product is $(\phi(x), f(x)) = \int_0^\pi \phi(x)f(x)dx$. We will be computing this inner product for several functions related to $\sin(nx)$ for $n > 0$ and $\cos(nx)$ for $n \geq 0$.

For starters note:

$$(\cos(0x), x^m) = \int_0^\pi x^m dx = \frac{\pi^{m+1}}{m+1}$$

So we may **assume** $n > 0$ for our **remaining** computations. For starters when $n \neq 0$ we have

$$(\cos(nx), 1) = \int_0^\pi cos(nx)dx = 0$$

and

$$(\sin(nx), 1) = \int_0^\pi \sin(nx)dx = \left\{ \begin{array}{ll} \frac{2}{n} & n = 2k - 1 \\ 0 & n = 2k \end{array} \right\}$$

Now note integration by parts gives:

$$(\cos(nx), x^m) = \int_0^\pi x^m \cos(mx)dx$$

$$= \frac{x^m}{n}\sin(nx)|_0^\pi - \frac{m}{n}\int_0^\pi x^{m-1}\sin(nx)dx = -\frac{m}{n}(x^{m-1}, \sin(nx))$$

and

$$(\sin(nx), x^m) = \int_0^\pi x^m \sin(nx)dx = -\frac{x^m}{n}\cos(nx)|_0^\pi + \frac{m}{n}\int_0^\pi x^{m-1}\cos(nx)dx$$

$$= \left\{ \begin{array}{ll} \frac{\pi^m}{n} + \frac{m}{n}(x^{m-1}, \cos(nx)) & n = 2k-1 \\ \frac{-\pi^m}{n} + \frac{m}{n}(x^{m-1}, \cos(nx)) & n = 2k \end{array} \right\}.$$

Now observe by recursion theses computations give explicit numbers for $(\sin(nx), x^m)$ and $(\cos(nx), x^m)$ for all n and m.

As an examples of this note:

$$(\cos(nx), x) = \frac{-1}{n}(1, \sin(nx)) = \left\{ \begin{array}{ll} -\frac{2}{n^2} & n = 2k-1 \\ 0 & n = 2k \end{array} \right\}$$

and

$$(\sin(nx), x) = \left\{ \begin{array}{ll} \frac{\pi}{n} + \frac{1}{n}(1, \cos(nx)) = \frac{\pi}{n} & n = 2k-1 \\ \frac{-\pi}{n} + \frac{1}{n}(1, \cos(nx)) = \frac{-\pi}{n} & n = 2k \end{array} \right\}.$$

Similarly

$$(\cos(nx), x^2) = \frac{-2}{n}(x, \sin(nx)) = \left\{ \begin{array}{ll} \frac{-2\pi}{n^2} & n = 2k-1 \\ \frac{2\pi}{n^2} & n = 2k \end{array} \right\}.$$

and

$$(\sin(nx), x^2) = \left\{ \begin{array}{ll} \frac{\pi^2}{n} + \frac{2}{n}(x, \cos(nx)) = \frac{\pi^2}{n} + \frac{-4}{n^3} & n = 2k-1 \\ \frac{\pi^2}{n} + \frac{2}{n}(x, \cos(nx)) = \frac{-\pi^2}{n} & n = 2k \end{array} \right\}$$

Our actual problem in fact involves certain related inner products, let $a_n(f) = \frac{2}{\pi}(f(x), \cos(nx))$ and $b_n(f) = \frac{2}{\pi}(f(x), \sin(nx))$ for any f(x).

To use the above formulas efficiently note a_n and b_n linear namely, $a_n(cf + g) = ca_n(f) + a_n(g)$. This gives us in particular that $a_n(cx^2 + dx^2 + ef) = ca_n(x^2) + da_n(x) + ea_n(1)$. Similarly for b_n.

p. 56. 4: (a)From above when $n \neq 0$

$$a_n(x^2) = \frac{2}{\pi} \left\{ \begin{array}{ll} \frac{-2\pi}{n^2} & n = 2k-1 \\ \frac{2\pi}{n^2} & n = 2k \end{array} \right\} = \left\{ \begin{array}{ll} \frac{-4}{n^2} & n = 2k-1 \\ \frac{4}{n^2} & n = 2k \neq 0 \end{array} \right\}$$

For $n = 0$ we have $a_0 = \frac{2\pi^3}{3}$.

$$f \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(nx)$$

(b)

$$b_n(x^2) = \frac{2}{\pi} \left\{ \begin{array}{ll} \frac{\pi^2}{n} + \frac{-4}{n^3} & n = 2k-1 \\ \frac{\pi^2}{n} & n = 2k \end{array} \right\} = \left\{ \begin{array}{ll} \frac{2\pi}{n} - \frac{8}{n^3\pi} & n = 2k-1 \\ \frac{-2\pi}{n} & n = 2k \neq 0 \end{array} \right\}$$

$$f \sim 2\pi^2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n\pi} - 2\frac{1 + (-1)^{n+1}}{(n\pi)^3} \right] \sin(nx)$$

p. 56. 5:

$$b_n(x(\pi - x)) = b_n(\pi x - x^2) = \pi b_n(x) - b_n(x^2)$$

$$= \frac{2}{\pi}\left(\pi \left\{ \begin{array}{ll} \frac{\pi}{n} & n = 2k-1 \\ \frac{-\pi}{n} & n = 2k \end{array} \right\} - \left\{ \begin{array}{ll} \frac{\pi^2}{n} + \frac{-4}{n^3} & n = 2k-1 \\ \frac{-\pi^2}{n} & n = 2k \end{array} \right\}\right)$$

$$= \frac{2}{\pi} \left\{ \begin{array}{ll} \frac{\pi^2}{n} & n = 2k-1 \\ \frac{-\pi^2}{n} & n = 2k \neq 0 \end{array} \right\} - \frac{2}{\pi} \left\{ \begin{array}{ll} \frac{\pi^2}{n} + \frac{-4}{n^3} & n = 2k-1 \\ \frac{-\pi^2}{n} & n = 2k \neq 0 \end{array} \right\}$$

$$= \frac{2}{\pi} \left\{ \begin{array}{ll} \frac{4}{n^3} & n = 2k-1 \\ 0 & n = 2k \neq 0 \end{array} \right\}$$

$$f \sim \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^3}$$

p. 56. 6: Note that

$$b_n(\sin(x)) = (\sin(x), \sin(nx)) = \frac{\pi}{2}(\phi_n, \phi_1) = \left\{ \begin{array}{ll} 0 & n \neq 1 \\ \frac{\pi}{2} & n = 1 \end{array} \right\}$$

So

$$f \sim \frac{2}{\pi}\frac{\pi}{2} \sin(x) = \sin(x).$$

Note: For the next four problems we'll be looking for the Fourier series of various functions. This means expressing nicely $f \sim \sum (f, \phi_n)\phi_n$ where the ϕ_n are the elements of the orthonormal family

$$\{\phi_0 = \frac{1}{\sqrt{2\pi}}\} \cup \{\phi_{2k} = \frac{1}{\sqrt{\pi}} \sin(kx)\}_{k=1}^{\infty} \cup \{\phi_{2k-1} = \frac{1}{\sqrt{\pi}} \cos(kx)\}_{k=1}^{\infty}.$$

To do this observe $(f, \phi_0) = \frac{1}{\sqrt{2\pi}}(f, 1)$, $(f, \phi_{2k}) = \frac{1}{\sqrt{\pi}}(f, \sin(kx))$, and $(f, \phi_{2k-1}) = \frac{1}{\sqrt{\pi}}(f, \cos(kx))$. So we will always compute these then plug into the above formula.

p. 65. 1:

$$(f, 1) = \int_{-\pi}^{\pi} f(x)dx = \frac{-\pi}{2} \int_{-\pi}^{0} dx + \frac{\pi}{2} \int_{0}^{\pi} dx = 0$$

$$(f, \sin(kx)) = \frac{-\pi}{2} \int_{-\pi}^{0} \sin(kx)dx + \frac{\pi}{2} \int_{-\pi}^{0} \sin(kx)dx$$

$$= \frac{\pi}{2k}(1 - \cos(k\pi)) + \frac{-\pi}{2k}(\cos(k\pi)) - 1) = 2\frac{\pi}{2k}(1 - \cos(k\pi))[= \left\{ \begin{array}{ll} \frac{2\pi}{k} & k = 2l-1 \\ 0 & n = 2l \end{array} \right\}.$$

Similarly

$$(f, \cos(kx)) = \frac{-\pi}{2} \int_{-\pi}^{0} \cos(kx)dx + \frac{\pi}{2} \int_{0}^{\pi} \cos(kx)dx$$

$$= \frac{-\pi}{2k}0 + \frac{-\pi}{2k}0 = 0$$

So

$$f \sim \sum (f, \phi_n)\phi_n = 0 + \sum_{l=l}^{\infty} \frac{1}{\pi} \sin((2l-1)x)\frac{2\pi}{2l-1} = \sum_{l=l}^{\infty} 2\frac{\sin((2l-1)x)}{2l-1}$$

p. 66. 2: First note the function is

$$\left\{ \begin{array}{ll} \frac{2}{\pi}x + 2 & -\pi \ \leq x \leq 0 \\ 2 & 0 \leq x \leq \pi \end{array} \right\}.$$

So

$$(f, 1) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(\frac{2}{\pi}x + 2)dx +]2\int_{0}^{\pi} dx\right) = 3$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = \frac{1}{\pi}(\int_{-\pi}^{0}(\frac{2}{\pi}x + 2)\sin(kx)dx + 2\int_{0}^{\pi}\sin(kx)dx)$$

Then we have $\int_{-\pi}^{\pi}\sin(nx)dx = 0$ so

$$= \frac{2}{\pi^2}\int_{-\pi}^{0} x\sin(kx)dx$$

and so with integration by parts we arrive at

$$= \frac{2(-1)^{k+1}}{\pi k}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}(\int_{-\pi}^{0}(\frac{2}{\pi}x + 2)\cos(kx)dx + 2\int_{0}^{\pi}\cos(kx)dx)$$

The terms not multiplied by x above evaluate to 0 so we are left with

$$\frac{2}{\pi^2}(\int_{-\pi}^{0} x\cos(kx)dx$$

which by integration by parts gives us

$$0 + 0 + \frac{2}{\pi^2}(\frac{\cos(kx)}{k^2}) \mid_{-\pi}^{0} = \frac{2}{\pi^2 k^2}((-1)^k + 1)$$

p. 66. 4:

$$(f, 1) = \int_{-\pi}^{\pi} e^{ax}dx = \frac{2}{a}\frac{(e^{\pi a} - e^{-\pi a})}{2} = \frac{2\sinh(a\pi)}{a}$$

$$(f, \cos(kx)) + i(f, \sin(kx)) = \int_{-\pi}^{\pi} e^{ax}\cos(kx)dx + \int_{-\pi}^{\pi} e^{ax} i\sin(kx)dx = \int_{-\pi}^{0} e^{ax}e^{ikx}dx$$

$$= \int_{-\pi}^{0} e^{(a+ik)x}dx = \frac{1}{a+ik}(e^{(a+ik)\pi} - e^{-(a+ik)pi})$$

$$= \frac{a-ik}{(a-ik)(a+ik)} \left(e^{a\pi}(\cos(k\pi) + i\sin(k\pi) - e^{-a\pi}(\cos(k\pi) - i\sin(k\pi)\right)$$

$$= \frac{a-ik}{a^2+k^2}((e^{a\pi} - e^{-a\pi})\cos(k\pi)) = \frac{2}{a^2+k^2}(a-ik)((\sinh(a\pi))(-1)^k)$$

$$= \frac{2a}{a^2+k^2}(\sinh(a\pi))(-1)^k + i\left(\frac{-2k}{a^2+k^2}(\sinh(a\pi))(-1)^k\right)$$

So

$$f \sim \sum (f,\phi_n)\phi_n$$

$$= \frac{1}{2\pi}\frac{2\sinh(a\pi)}{a} + \frac{1}{\pi}\sum_{k=l}^{\infty}\frac{2a}{a^2+k^2}(\sinh(a\pi))(-1)^k\cos(kx) + \frac{-2k}{a^2+k^2}(\sinh(a\pi))(-1)^k\sin(kx)$$

$$= \frac{2\sinh(a\pi)}{\pi}\left[\frac{1}{2a} + \sum_{k=l}^{\infty}\frac{(-1)^k}{a^2+k^2}(a\cos(kx) - k\sin(kx))\right]$$

p. 66. 5: This problem is an immediate consequence if the previous since

$$(\sinh(ax),\phi_k) = (\frac{e^{ax}-e^{-ax}}{2},\phi_k) = \frac{1}{2}((e^{ax},\phi_k) - (e^{-ax},\phi_k))$$

So form the previous problem (or use the previous problem and that $\sinh(x)$ is odd).

$$(\sinh(ax), \cos(kx) + i\sin(kx)) = i\left(\frac{-2k}{a^2+k^2}(\sinh(a\pi))(-1)^k\right)$$

Also by oddness

$$f \sim \sum (f,\phi_n)\phi_n = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{-2k}{a^2+k^2}(\sinh(a\pi))(-1)^k\sin(kx)\right)$$

p. 80. 2: Since on $[0,\pi]$ we have $\sin(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}$ (p. 79 or result 1 in the next section) evaluating at zero

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

or

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Evaluatin at $\frac{\pi}{2}$ we have

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

or

$$-\frac{\pi}{4} + \frac{1}{2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$