Math 110, Homework Solution Set 2
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For the next three problems I will be using the new interval $[-c, c]$. Here the orthonormal set corresponding to the Fourier series on $[-\pi, \pi]$ becomes $\{\phi_k^c\} = \{\sqrt{\frac{\pi}{c}}\phi_k(\frac{\pi}{c}x)\}$ on $[-c, c]$. So our new series is $f \sim \sum(f, \phi_n^c)\phi_k^c$ where

$$(f, \phi_k^c)_{new} = \sqrt{\frac{\pi}{c}}\int_{-c}^{c} f(x)\phi_k(\frac{\pi}{c}x)dx = \sqrt{\frac{c}{\pi}}\int_{-\pi}^{\pi} f(\frac{c}{\pi}u)\phi_k(u)du = \sqrt{\frac{c}{\pi}}(\phi_k, f(\frac{c}{\pi}x))_{old}$$

and so

$$f \sim \frac{1}{2c}(f,1)_{new} + \frac{1}{c}\sum(f(x), \sin(\frac{n\pi c}{c}))_{new}\sin(\frac{n\pi x}{c}) + \frac{1}{c}\sum(f(x), \cos(\frac{n\pi x}{c}))_{new}\cos(\frac{n\pi x}{c})$$

$$= \sum(f, \phi_n^c)\phi_k^c = \sum\sqrt{\frac{c}{\pi}}(\phi_k, f(\frac{c}{\pi}x))_{old}\sqrt{\frac{\pi}{c}}\phi_k(\frac{\pi}{c}x)$$

$$= \sum(\phi_k, f(\frac{c}{\pi}x))_{old}\phi_k(\frac{\pi}{c}x)$$

$$= \frac{1}{2\pi}(f(\frac{cx}{\pi}), 1)_{old} + \frac{1}{\pi}\sum(f(\frac{cx}{\pi}), \sin(x))_{old}\sin(\frac{n\pi cx}{c}) + \frac{1}{\pi}\sum(f(\frac{cx}{\pi}), \cos(x))_{old}\cos(\frac{n\pi x}{c})$$

.

This last formula is what I'll use.

Also observe sometimes I change the definition of the functions in these problems to use result one in the next section to replace $\sim$ with $=$.

**p. 85. 1:**

$$f = \left\{\begin{array}{ll} 0 & -3 \le x < 0 \\ 1 & 0 < x \le 3 \\ \frac{1}{2} & x = 0 \end{array}\right\}$$

now note

$$f(\frac{3}{\pi}x) = \left\{\begin{array}{ll} 0 & -\pi \le x < 0 \\ 1 & 0 < x \le \pi \\ \frac{1}{2} & x = 0, 3, -3 \end{array}\right\}$$

So

$$(\sin(nx), f(\frac{c}{\pi}x))_{old} = \int_0^{\pi} \sin(nk)dx = \frac{-cos(nx)}{n}|_0^{\pi} = \frac{(-1)^{n+1}+1}{n}.$$

Similarly

$$(\cos(nx), f(\frac{c}{\pi}x))_{old} = \int_0^{\pi} \cos(nk)dx = \frac{\sin(nx)}{n}|_0^{\pi} = 0.$$

and

$$(1, f(\frac{c}{\pi}x))_{old} = \pi$$

So

$$f \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi} \frac{\sin((2k-1)x)}{2k-1}$$

In fact

$$f = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi} \frac{\sin((2k-1)x)}{2k-1}$$

by result one in the next section - since $f = \frac{1}{2}$ at $0, 3, -3$.

**p. 85 5.(a):** By the refered to problem

$$(\pi x, \sin(nx)) = \sqrt{\pi}(x, \phi_{2n}) = \sqrt{\pi} \frac{1}{\sqrt{\pi}} \frac{2(-1)^n}{n}$$

The other coeefecients are zero since $x$ is odd. So

$$f \sim \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x).$$

**p. 85. 9:** Our function is the periodic extention of

$$f = \left\{ \begin{array}{ll} -1 & -c < x < 0 \\ 1 & 0 < x < c \\ 0 & x = 0, c, -c \end{array} \right\}$$

(Where the values I choose at $lc$ are provided by looking at result one telling us where the series converges point wise). Note further that series is the odd extention of

$$f = \left\{ \begin{array}{ll} 1 & 0 < x < c \\ 0 & x = 0, c \end{array} \right\}$$

So the needed Fourier series is given by the Fourier sine series of this function squished to this interval. From 1(b) in section 14 on $[0, \pi]$ we have

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$$

Hence our new series is

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(\frac{(2n-1)\pi}{c} x)}{2n-1}$$

Which by the next result one and the choices made is in fact equal to the given function at each point.

**p. 118. 1:** From problem 4(a) on p. 56:

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx 0 < x < \pi$$

Extract the coefficients from this Fourier series and transplant them into (14) on p. 113 to obtain

$$u(x,t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-n^2 kt} \cos nx$$

**p. 118. 2:** If $u(x,y) = X(x)Y(y)$, then

$$\begin{aligned} u_{xx}(x,y) &= -u_{yy}(x,y) \\ X''(x)Y(y) &= X(x)Y(y) \\ \frac{X''(x)}{X(x)} &= -\frac{Y''(y)}{Y(y)} \end{aligned}$$

The left-hand side is a function of $x$ and the right-hand side is a function of $y$, hence they are both equal to a constant, say $-\lambda$. This gives

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad Y''(y) - \lambda Y(y) = 0$$

Since we are looking for interesting solutions, we can assume that $X, Y \neq 0$. Hence

$$\begin{aligned} 0 = u_x(0,y) = X'(0)Y(y) &\Rightarrow X'(0) = 0 \\ 0 = u_x(\pi,y) = X'(\pi)Y(y) &\Rightarrow X'(\pi) = 0 \\ 0 = u(x,0) = X(x)Y(0) &\Rightarrow Y(0) = 0 \end{aligned}$$

$\lambda = 0$**:** Then $X(x) = ax + b$ and $X'(x) = a$, so $X'(0) = 0$ implies $a = 0$. So we can choose $X_0(x) = 1$ up to constant multiple. Solving for $Y$, we find $Y(y) = cy + d$, and $Y(0) = 0$ forces $d = 0$. Hence $Y_0(y) = y$ up to constant multiple, and $u_0(x,y) = y$.

$\lambda < 0$**:** Then $X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$ and $X'(x) = c_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - c_2\sqrt{\lambda}e^{\sqrt{-\lambda}x}$. So $0 = X'(0) = (c_1 - c_2)\sqrt{\lambda}$ implies $c_1 = c_2$. But $0 = X'(\pi) = c_1\sqrt{\lambda}(e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}$. But $e^{\sqrt{\lambda}\pi} > 1$ and $e^{-\sqrt{\lambda}\pi} < 1$ so this would force $c_1 = 0$ and hence $c_2 = 0$. The result is $X(x) = 0$, which gives the trivial solution $u(x,y) = 0$.

$\lambda > 0$**:** Then $X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(-\sqrt{\lambda}x)$ and $X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{-\lambda}x)$. So $0 = X'(0) = c_2\sqrt{\lambda}$ implies $c_2 = 0$. Now use $0 = X'(\pi) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)$ to conclude either $c_1 = 0$ (which would again yield the trivial solution) or $\sqrt{\lambda}\pi = n\pi$ for any integer $n$. Hence $X_n(x) = \cos nx$. Note that we may restrict $n$ to be a positive as cos is an even function and $n = 0$ just gives $X_0(x) = 1$, which we already have.
Now $Y(y) = c_3 e^{\sqrt{\lambda}y} + c_4 e^{-\sqrt{\lambda}y} = c_3 e^{ny} + c_4 e^{-ny}$. The boundary condition $Y(0) = 0$ implies $c_4 = -c_3$ and $Y_n(y) = \sinh ny$ up to constant multiple. Hence $u_n(x,y) = \cos nx \sinh ny$.

So the general solution is $u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos nx \sinh ny$. Comparing the remaining boundary condition

$$f(x) = u(x,2) = 2A_0 + \sum_{n=1}^{\infty} A_n \cos nx \sinh 2n$$

with the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

for

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

gives

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} f(x)\,dx \quad \text{and} \quad A_n = \frac{2}{\pi\sinh 2n}\int_0^{\pi} f(x)\cos nx\,dx \text{ for } n \geq 1$$

**p. 118. 3: a:**

$$\begin{aligned} X''(x)T(t) - xtX(x)T''(t) &= 0 \\ \frac{X''(x)}{xX(x)} &= \frac{tT''(t)}{T(t)} = -\lambda \end{aligned}$$

since the LHS is a function of $x$ and the RHS is a function of $t$ only.
Hence $X''(x) + \lambda xX(x) = 0$ and $tT''(t) + \lambda T(t) = 0$.

**c:** This is not a separable equation as each of the three terms contains both $x$ and $t$.

**p. 119. 7:** This is trivial. Just use the linearity of differentiation.

**p. 126. 1:** Using equation (7) on p. 124, $F(x) = A\sin \pi x$, and the solution is

$$y(x,t) = \frac{F(x+at) + F(x-at)}{2} = \frac{A(\sin(\pi x + \pi at) + \sin(\pi x - \pi at))}{2} = A\sin(\pi x)\cos(\pi at)$$

This indeed satisfies:

$$\begin{aligned} 0 &= y(0,t) = A\sin 0\cos(\pi at) \\ 0 &= y(1,t) = A\sin\pi\cos(\pi at) \\ 0 &= y_t(x,0) = -aA\sin(\pi x)\sin 0 \\ A\sin(\pi x) &= y(x,0) = A\sin(\pi x)\cos 0 \end{aligned}$$

Also, $y(x,t)$ is clearly continuous in $x$ and $t$ and $y_t(x,t) = -aA\sin(\pi x)\cos(\pi at)$ is continuous in $t$.

**p. 126. 2:** This is almost exactly the same as the previous problem.

**p. 127. 4:** If $y(x,t) = X(x)T(t)$, then

$$\begin{aligned} y_{tt}(x,t) &= a^2 y_{xx}(x,t) \\ X(x)T''(t) &= a^2X''(x)T(t) \\ \frac{X''(x)}{X(x)} &= \frac{T''(t)}{a^2T(t)} \end{aligned}$$

The left-hand side is a function of $x$ and the right-hand side is a function of $t$, hence they are both equal to a constant, say $-\lambda$. This gives

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T''(t) + \lambda a^2 T(t) = 0$$

Since we are looking for interesting solutions, we can assume that $T, X \neq 0$. Hence

$$\begin{aligned} 0 = y(0,t) = X(0)T(t) &\Rightarrow X(0) = 0 \\ 0 = y(c,t) = X(c)T(t) &\Rightarrow X(c) = 0 \\ 0 = y_t(x,0) = X(x)T'(0) &\Rightarrow T'(0) = 0 \end{aligned}$$

**p. 127. 5:** Consider the following three cases:

**$\lambda < 0$:** Let $\alpha = \sqrt{-\lambda}$. The solutions of $X''(x) + \lambda X(x) = 0$ are then $X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Now

$$\begin{aligned} X(0) = 0 &\Rightarrow c_2 = -c_1 \\ X(c) = 0 &\Rightarrow c_1(e^{\alpha c} - e^{-\alpha c}) = 0 \end{aligned}$$

Since $e^{\alpha c} - e^{-\alpha c} = e^{\alpha c} - 1/e^{\alpha c} \neq 0$ for $c \neq 0$, $c_1 = 0$. So we only get the trivial solution $X_0(x) = 0$.

**$\lambda = 0$:** Then $X''(x) = 0$, and hence $X(x) = ax + b$. $X(x) = 0$ implies $b = 0$ and $X(c) = 0$ implies $a = 0$. Again, this is just the trivial solution.

**$\lambda > 0$:** Let $\alpha = \sqrt{\lambda}$. Then the solutions of $X''(x) + \lambda X(x) = 0$ are of the form $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$. From the initial conditions:

$$\begin{aligned} X(0) = 0 &\Rightarrow c_3 = 0 \\ X(c) = 0 &\Rightarrow c_4 \sin \alpha c = 0 \end{aligned}$$

So either $c_4 = 0$, which is again the trivial solution, or $\alpha c = n\pi$. Then $\alpha = n\pi/c$, and $X_n(x) = \sin(n\pi x/c)$ up to constant multiple. Since we are ignoring the constant $c_4$ anyway, we may as well assume $n = 0, 1, 2, \ldots$, as negative values of $n$ only multiply the solution by $-1$ (sin is odd).

So the solutions are $X_n(x) = \sin \frac{n\pi x}{c}$ for $n = 0, 1, 2, \ldots$ with eigenvalues $\lambda = n^2\pi^2/c^2$. (Notice that the trivial solution is included.)

**p. 135. 2:** Note that $f(x) = \sin x$ is already a Fourier sine series with coefficients $b_1 = 1$ and $b_n = 0$ for $n \neq 1$. By equation (5) on p. 131, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 kt} \sin(nx) = e^{-kt} \sin x$$

**p. 135. 3:** We need to solve the system of PDEs:

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) \\ u(0,t) &= 0 \\ u(\pi,t) &= u_0 \\ u(x,0) &= f(x) \end{aligned}$$

We know from Example 1 that

$$\begin{aligned} v_t(x,t) &= k v_{xx}(x,t) \\ v(0,t) &= 0 \\ v(\pi,t) &= 0 \\ v(x,0) &= f(x) \end{aligned}$$

From Example 2,

$$\begin{aligned} w_t(x,t) &= k w_{xx}(x,t) \\ w(0,t) &= 0 \\ w(\pi,t) &= u_0 \\ w(x,0) &= 0 \end{aligned}$$

Add the corresponding equations

$$\begin{aligned} v_t(x,t) + w_t(x,t) &= k(v_{xx}(x,t) + w_{xx}(x,t)) \\ v(0,t) + w(0,t) &= 0 \\ v(\pi,t) + w(\pi,t) &= u_0 \\ v(x,0) + w(x,0) &= f(x) \end{aligned}$$

Hence $v + w$ is a solution of this problem.