Math 110, Homework Solution Set 3

Imre Tuba

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p. 140. 1: Let $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n\pi x)$. Then the equation becomes

$$\sum_{n=1}^{\infty} B_n'(t)\sin(n\pi x) = \sum_{n=1}^{\infty} B_n(t)\left(-(n\pi)^2\sin(n\pi x)\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}p(t)}{n}\sin(n\pi x)$$

After matching the coefficients of $\sin(n\pi x)$ on the two sides, we get

$$B_n'(t) + n^2\pi^2 B_n(t) = \frac{(-1)^{n+1}p(t)}{n}$$

The first two boundary conditions are already satisfied as $\sin 0 = 0$ and $\sin n\pi = 0$. The last boundary condition turns into

$$\sum_{n=1}^{\infty} B_n(0)\sin(n\pi x) = 0$$

which forces $B_n(0) = 0$ (a Fourier series is 0 if and only if all its coefficients are 0). Multiply the first order ODE above by the integrating factor $e^{\int n^2\pi^2\,dt} = e^{n^2\pi^2 t}$ to obtain

$$\begin{aligned} e^{n^2\pi^2 t}B_n'(t) + n^2\pi^2 e^{n^2\pi^2 t}B_n(t) &= \frac{2(-1)^{n+1}}{n\pi}p(t)e^{n^2\pi^2 t} \\ \left(e^{n^2\pi^2 t}B_n(t)\right)' &= \frac{2(-1)^{n+1}}{n\pi}p(t)e^{n^2\pi^2 t} \end{aligned}$$

Solve by integrating both sides:

$$\begin{aligned} e^{n^2\pi^2 t}B_n(t) &= \int \frac{2(-1)^{n+1}}{n\pi}p(t)e^{n^2\pi^2 t}\,dt \\ e^{n^2\pi^2 t}B_n(t) &= \frac{2(-1)^{n+1}}{n\pi}\int_0^t p(\tau)e^{n^2\pi^2\tau}\,d\tau + c \end{aligned}$$

where we used $\int f(x)\,dx = \int_0^x f(u)\,du + c$. Now use the boundary condition $B_n(0) = 0$ to conclude $c = 0$. Hence

$$B_n(t) = \frac{2(-1)^{n+1}}{n\pi}e^{-n^2\pi^2 t}\int_0^t p(\tau)e^{n^2\pi^2\tau}\,d\tau = \frac{2(-1)^{n+1}}{n\pi}\int_0^t p(\tau)e^{n^2\pi^2(\tau-t)}\,d\tau$$

and

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)\int_0^t p(\tau)e^{n^2\pi^2(\tau-t)}\,d\tau$$

p. 141. 2: Just evaluate the integral in the solution:

$$\int_0^t a e^{n^2\pi^2(\tau-t)}\,d\tau = \left.\frac{a e^{n^2\pi^2(\tau-t)}}{n^2\pi^2}\right|_0^t = a\frac{1-e^{-n^2\pi^2 t}}{n^2\pi^2}$$

and substitute this into the solution of the previous problem.

p. 148. 1.: The equations are:

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0 \\ u_x(0,y) = u_x(\pi,y) &= 0 \\ u(x,0) &= 0 \\ u(x,\pi) &= f(x) \end{aligned}$$

This is another separable differential equation leading to the two ODEs

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \qquad & X'(0) = X'(\pi) = 0 \\ Y''(y) - \lambda Y(y) &= 0 \qquad & Y(0) = 0 \end{aligned}$$

These are easy to solve along the lines of problem 2 on p. 118 and give the results $X_0(x) = 1$, $X_n(x) = \cos nx$ and $Y_0(y) = y$, $Y_n(y) = \sinh ny$ for $n \geq 1$ up to constant multiple.

Hence $u(x,y) = A_0 y + \sum_{n=1}^{\infty} \cos nx \sinh ny$. Now use the last boundary condition and the Fourier series of $f(x)$:

$$u(x,\pi) = A_0\pi + \sum_{n=1}^{\infty} \cos nx \sinh n\pi = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

for

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Now compare coeffecients and rejoice when you get the solution in the book.

p. 157. 7: A string is stretched over $[0,\pi]$ and is initially at rest with $y = f(x)$. With air resistance the differential equation to solve is

$$y_{tt} = y_{xx} - 2\beta y_t.$$

Separating variables gives

$$\frac{X_{xx}}{X} = \frac{T_{tt} + 2\beta T_t}{T} = -\lambda$$

Using the boundary conditions $X(0) = X(\pi) = 0$ we get $X(x) = c_n \sin(nx)$. So $\lambda = n^2$.

Now letting $\alpha_n = \sqrt{4n^2 - \beta^2}$ the linear O.D.E.

$$T_{tt} + 2\beta T_t + n^2 T$$

has the solutions $T(t) = e^{-\beta t}(l_n \sin(\alpha_n t) + k_n \cos(\alpha_n t))$. To find its boundary conditions note $T_t(0) = (-\beta k_n + l_n) = 0$ giving $T(t) = c_n e^{-\beta}(\frac{\beta}{\alpha_n}\sin(\alpha_n) + \cos(\alpha_n))$.

So our formal solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta t}(\frac{\beta}{\alpha_n}\sin(\alpha_n t) + \cos(\alpha_n t))\sin(nx).$$

Note assuming $f(x)$ has a Fourier expansion $y_t(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ gives

$$y(x,0) = \sum_{n=1}^{\infty} cn \sin(nx) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

So $c_n = b_n$ and the formal solution is:

$$y(x,t) = \sum_{n=1}^{\infty} b_n e^{-\beta t}(\frac{\beta}{\alpha_n} \sin(\alpha_n t) + \cos(\alpha_n t) \sin(nx).$$

p. 157. 12: Explore $y_{tt} = a^2 y_{xx} + Ax\sin(\omega t)$ with $y(0,t) = y(c,t) = 0$ and $y(x,0) = y_t(x,0) = 0$.

The assume technique is needed here, i.e. from $y(0,t) = y(c,t) = 0$ we assume that a solution is in the form $\sum_{n=1}^{\infty} B_n(t) sin(\frac{n\pi}{c}x)$. Now we would like to plug this in to the equation, but to do so we must recall $x = \frac{2\pi}{c}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(\frac{\pi n}{c}x)$. Plugging in this and our assumed solution form now gives us.

$$\sum_{n=1}^{\infty}(\frac{d^2}{dt^2}B_n + (\frac{n\pi}{c})^2 B_n + \frac{2\pi}{c}\frac{(-1)^n}{n}\sin(\omega t))\sin(\frac{na\pi}{c}x) = 0$$

Now observe from the text that the solutions to this O.D.E are

$$B_n(t) = \frac{2(-1)^n c^2}{c^2\omega^2 n - n^2\pi^2 a^2}(\frac{\omega c}{n\pi a}\sin(\frac{n\pi a}{c}t) - \sin(\omega t))$$

when $\omega \neq \frac{n\pi a}{c}$ and

$$B_n(t) = \frac{2(-1)^n c}{2n^2\pi a}(\frac{1}{a}\sin(\omega t) - \sin(\omega t))$$

When $\omega = \frac{n\pi a}{c}$.

So the solution that increase in size linearly with time occur only at the resonance values $\omega = \frac{n\pi a}{c}$. (Note we've also found the solution.)

p. 176. 1: **a:** Note that if $x = e^s$, then $xX'(x) = e^s X'(e^s) = \frac{dX(e^s)}{ds}$ and $x(xX'(x))' = e^s(e^s X'(e^s))' = \frac{d^2X(e^s)}{ds^2}$. So multiply the equation by x and substitute $x = e^s$ to get

$$\frac{d^2X}{ds^2} + \lambda X = 0 \qquad (0 < s < \ln b)$$

where X is a function of s. When $x = 1$, $s = 0$ and when $x = b$, $s = \ln b$ and it is at these two points that $X = 0$.

We already know the solution of a Sturm-Liouville problem with such boundary conditions. Let $\alpha_n = n\pi/\ln b$ for $n = 1, 2, \ldots$. Then the eigenvalues are $\lambda_n = \alpha_n^2$ and the eigenfunctions are $X_n(s) = \sin\alpha_n s = \sin(\alpha_n \ln x)$.

b: We need to show

$$\int_1^b X_m(x)X_n(x)p(x)\,dx = \int_1^b \sin\left(\frac{m\pi}{\ln b}\ln x\right)\sin\left(\frac{n\pi}{\ln b}\ln x\right)\frac{1}{x}\,dx = 0$$

when $m \neq n$. Substitute $s = (\pi/\ln b)\ln x$:

$$\int_0^\pi \sin ms \sin ns \, ds = 0$$

for $m \neq n$ by (9) on p. 175.

p. 184. 2: Here we are dealing with

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad hX(1) + X'(1) = 0$$

with $h \geq 0$, so $a_1 a_2 = 0 \leq 0$ and $b_1 b_2 = h \geq 0$; and by the lemma there are no negative eigen-values ($\lambda \geq 0$).

Now for $\lambda = 0$ we have that $X(x) = ax + b$ giving

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X(0) \\ hX(1) + X'(1) \end{bmatrix} = \begin{bmatrix} b \\ ha + hb + a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (1+h) & h \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Now note that for this to have any non trivial solutions

$$\begin{bmatrix} 0 & 1 \\ (1+h) & h \end{bmatrix}$$

would need to have a null space so be non-invertible. But since $h > 0$

$$\det \begin{bmatrix} 0 & 1 \\ (1+h) & h \end{bmatrix} = -(1+h) \neq 0.$$

So it is invertible and there are no nontrivial solutions with $\lambda = 0$.

Similarly for $\lambda > 0$ we can call $\lambda = \alpha^2$ and we have

$$X(x) = a\sin(\alpha x) + b\cos(\alpha x).$$

So

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X(0) \\ hX(1) + X'(1) \end{bmatrix} = \begin{bmatrix} b \\ ah\sin(\alpha) + bh\cos(\alpha) + a\alpha\cos(\alpha) - b\alpha\sin(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ h\sin(\alpha) + \alpha\cos(\alpha) & h\cos(\alpha) - \alpha\sin(\alpha) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Once again we are detecting the invertabilty of this matrix so we look at it's determinant an note

$$\det \begin{bmatrix} 0 & 1 \\ h\sin(\alpha) + \alpha\cos(\alpha) & h\cos(\alpha) - \alpha\sin(\alpha) \end{bmatrix} = -(h\sin(\alpha) + \alpha\cos(\alpha)).$$

So we have non trivial solution exact for $\alpha > 0$ when $h\sin(\alpha x) + \alpha\cos(\alpha) = 0$ or $\frac{-\alpha}{h} = \tan(\alpha)$. Looking at the graph these are indexed by an increasing sequence of number α_n.

Once again the null space is $b = 0$ so our non-normalized eigen-functions are $X(x) = a\sin(\alpha_n x)$.

To normalize we need a such that

$$1 = (a\sin(\alpha_n x), a\sin(\alpha_n x)) = a^2 \int_0^1 (\sin(\alpha_n x))^2 dx.$$

As above

$$a = \frac{1}{\sqrt{\frac{1}{2} + \frac{-1}{4\alpha_n}(\sin(2\alpha_n))}}$$

if $\frac{1}{2} + \frac{-1}{4\alpha_n}(\sin(2\alpha_n)) > 0$,

Now looking at the intersection of $\frac{-\alpha}{h}$ and $\tan(\alpha)$ we see that the smallest α is once again $\alpha_1 \geq \frac{\pi}{2}$. So this expression makes sense and finishes the problem.

p. 185. 4: Here we are dealing with

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad X(1) - X'(1) = 0.$$

Now for $\lambda = 0$ we have that

$$X(x) = ax + b$$

So

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) - X'(1) \end{bmatrix} = \begin{bmatrix} b \\ a + b - a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Now note that for this to have any non-trivial solutions

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

would need to have a null space so be non-invertible.

And in fact

$$\det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

Looking at the matrix the null space is where $b = 0$ and a is arbitrary.

So $X(x) = aX$ is a solution. To normalize we need a such that

$$1 = (ax, ax)) = a^2 \int_0^1 x^2 dx = a^2 \frac{1}{3}$$

Or $a = \sqrt{3}$.

Similarly for $\lambda > 0$ we can call $\lambda = \alpha^2$ and we have

$$X(x) = a\sin(\alpha x) + b\cos(\alpha x)$$

So

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) - X'(1) \end{bmatrix} = \begin{bmatrix} b \\ a\sin(\alpha) + b\cos(\alpha) - a\alpha\cos(\alpha) + b\alpha\sin(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ (\sin(\alpha) - \alpha\cos(\alpha)) & (\cos(\alpha) + \alpha\sin(\alpha)) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Once again we are detecting the invertabilty of this matrix so we look at it's determinant an note

$$\det \begin{bmatrix} 0 & 1 \\ \sin(\alpha) - \alpha\cos(\alpha) & \cos(\alpha) + \alpha\sin(\alpha) \end{bmatrix} = \sin(\alpha) - \alpha\cos(\alpha).$$

So we have non trivial solution exact for $\alpha > 0$ where $h\sin(\alpha x) + \alpha\cos(\alpha) = 0$ or $\alpha = \tan(\alpha)$. Looking at the graph these are indexed by an increasing sequence of number α_n.

Once again the null-space is $b = 0$ so our non normalized eigen-functions are $X(x) = a\sin(\alpha_n x)$.

As in problem one to normalize we need to make sense out of

$$a = \frac{1}{\sqrt{\frac{1}{2} - \frac{1}{4\alpha_n}(\sin(2\alpha_n))}}.$$

Equivalently we need $\frac{1}{2} + \frac{-1}{4\alpha_n}(\sin(2\alpha_n)) > 0$.

Once again looking at the intersection of α and $\tan(\alpha)$ we see that the smallest α is once again $\alpha_1 \geq \frac{\pi}{2}$ (here since $\tan(x)$ is concave up in $(0, \frac{\pi}{2})$ and α is its tangent line a zero - so the first intersection will not occur in $(0, \frac{\pi}{2})$). So this expression makes sense and gives us the normalized eigen-functions when $\lambda > 0$.

For this problem we must also look into the $\lambda < 0$, since our corollary fails here.

The solution here is

$$X(x) = ae^{\alpha x} + be^{-\alpha x}.$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) - X'(1) \end{bmatrix} = \begin{bmatrix} a + b \\ (-a\alpha + a)e^{\alpha} + (b + \alpha b)e^{-\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ (1-\alpha)e^{\alpha} & (1+\alpha)e^{-\alpha} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 1 \\ (1-\alpha)e^{\alpha} & (1+\alpha)e^{-\alpha} \end{bmatrix} = (1+\alpha)e^{-\alpha} + (-1+\alpha)e^{\alpha}.$$

So there is a non trivial solution when $(1+\alpha)e^{-\alpha} + (-1+\alpha)e^{\alpha} = 0$ or $e^{alpha} - e^{-\alpha} = \alpha(e^{\alpha} + e^{-\alpha})$. Divide the last equation by 2 to get $\sinh\alpha = \alpha\cosh\alpha$.

This can't happen. The way to see this is to look at the derivatives of these functions. The derivative of the LHS is $\cosh\alpha$ and the derivative of the RHS is $\cosh\alpha + \alpha\sinh\alpha$. For $\alpha > 0$,

$$\cosh\alpha < \cosh\alpha + \alpha\sinh\alpha$$

hence the RHS increases faster than the LHS. When $\alpha = 0$, $\sinh\alpha = \cosh\alpha + \alpha\sinh\alpha\alpha\cosh\alpha$, but this can't happen ever again for $\alpha > 0$ because the RHS will always be larger.

p. 191. 2: Use the general formula

$$1 = \sum_{n=1}^{\infty} \phi_n(x) \int_0^c 1 \cdot \phi_n(x)\, dx$$

with $\phi_n(x) = \sqrt{\frac{2}{c}} \sin \alpha_n x$ and $\alpha_n = \frac{(2n-1)\pi}{2c}$.

$$\begin{aligned}
1 &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{c}} \sin \alpha_n x int_0^c \sqrt{\frac{2}{c}} \sin \alpha_n x\, dx \\
&= \sum_{n=1}^{\infty} \frac{2}{c} \sin \alpha_n x \left. \frac{-\cos \alpha_n x}{\alpha_n} \right|_0^c \\
&= \sum_{n=1}^{\infty} \frac{2}{c\alpha_n} \sin \alpha_n x \left(\cos 0 - \cos \frac{(2n-1)\pi}{2} \right) \\
&= \frac{2}{c} \sum_{n=1}^{\infty} \sin \alpha_n x \alpha_n
\end{aligned}$$

p. 304. 1: **a:**

$$\int_{-1}^{1} P_n(x)\, dx = \int_{-1}^{1} P_0(x) P_n(x)\, dx = 0$$

for $n = 1, 2, 3, \ldots$ by orthogonality of P_0 and P_n.

b:

$$\int_{-1}^{1} (Ax + B) P_n(x)\, dx = \int_{-1}^{1} (AP_0(x) + BP_1(x) P_n(x)\, dx$$

$$= A \int_{-1}^{1} P_1(x) P_n(x)\, dx + B \int_{-1}^{1} P_0(x) P_n(x) = 0$$

for $n = 2, 3, 4, \ldots$ by orthogonality of P_0 and P_1 to P_n.

p. 304. 2: All you need to do is evaluate six integrals. Here are two of them

$$\int_{-1}^{1} P_0(x) P_2(x)\, dx = \frac{1}{2} \int_{-1}^{1} 3x^2 - 1\, dx = \left. \frac{x^3}{2} - \frac{x}{2} \right|_{-1}^{1} = 0$$

$$\int_{-1}^{1} P_1(x) P_3(x)\, dx = \frac{1}{2} \int_{-1}^{1} 5x^4 - 3x^2\, dx = \left. \frac{x^5}{2} - \frac{x^3}{2} \right|_{-1}^{1} = 0$$

Now note that $P_m(x)P_n(x)$ is an odd function whenever one of m and n is odd and the other is even and the integral of an odd function from -1 to 1 is always 0. This takes care of the remaining four.

p. 304. 3: **a:** Note that $P_{2m}(x)P_{2n}(x)$ is an even function (product of two even functions), hence

$$int_0^1 P_{2m}(x) P_{2n}(x)\, dx = \frac{1}{2} int_{-1}^1 P_{2m}(x) P_{2n}(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{4n+1} & \text{if } m = n \end{cases}$$

by (15) on p. 304. Hence $||P_{2n}(x)|| = 1/\sqrt{4n+1}$ with respect to this inner product, and $\sqrt{4n+1}P_{2n}(x)$ is then a unit vector.

b: You can do the same as in (a). Note that the product of two odd functions is even.

p. 304. 4: Just integrate both sides of (10) from a to 1:

$$\begin{aligned}
(2n+1)\int_a^1 P_n(x)\,dx &= \int_a^1 P'_{n+1}(x) + P'_{n-1}(x)\,dx \\
\int_a^1 P_n(x)\,dx &= \frac{1}{2n+1}(P_{n+1}(1) - P_{n+1}(a) - P_{n-1}(1) + P_{n-1}(a)) \\
\int_a^1 P_n(x)\,dx &= \frac{1}{2n+1}(P_{n-1}(a) - P_{n+1}(a))
\end{aligned}$$

as $P_{n+1}(1) = P_{n-1}(1) = 1$.

p. 310. 1: Observe that $F(x)$ is the odd extension of the function $f(x) = 1$ to the interval $(-1,1)$. (The value at $x = 0$ is the average of the left limit and the right limit, exactly what we want for Fourier analysis.) We already know from (10) on p. 76 that

$$1 = \sum_{n=0}^{\infty}(P_{2n}(0) - P_{2n+2}(0))P_{2n+1}(x) \qquad (0 < x < 1)$$

Hence

$$F(x) = \sum_{n=0}^{\infty}(P_{2n}(0) - P_{2n+2}(0))P_{2n+1}(x) \qquad (-1 < x < 1)$$

Add $1/2 = 1/2P_0(x)$ to obtain

$$g(x) = \frac{1}{2} + \frac{F(x)}{2} = \frac{P_0(x)}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(P_{2n}(0) - P_{2n+2}(0))P_{2n+1}(x) \qquad (-1 < x < 1)$$

p. 310. 2: This was only recommended, but it's interesting.

a: $f(x)$ is continuous and piecewise smooth (the only point where it is not differentiable is $x = 0$), hence it $f(x)$ is equal to its Legendre series by the theorem on p. 308.

b: Note that $P_1(x)P_{2n+1}(x)$ is an even function. Hence

$$\begin{aligned}
A_{2n+1} &= \frac{2(2n+1)+1}{2}\int_{-1}^1 f(x)P_{2n+1}(x)\,dx \\
&= \frac{4n+3}{2}\int_0^1 xP_{2n+1}(x)\,dx = \frac{4n+3}{2}\int_0^1 P_1(x)P_{2n+1}(x)\,dx \\
&= \frac{4n+3}{4}\int_{-1}^1 P_1(x)P_{2n+1}(x)\,dx = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}
\end{aligned}$$

c: Just do the integration

$$A_n = \frac{2n+1}{2}\int_{-1}^1 f(x)P_n(x)\,dx = \frac{2n+1}{2}\int_0^1 xP_n(x)\,dx$$

for $n = 0, 2, 4$.