# Math 142 Exam 1 Solutions

Feb 23, 2015

1. (10 pts) Give an example of a function to which the Intermediate Value Theorem does not apply on the interval $-1 \le x \le 1$.

   The IVT is about functions that are continuous over a closed and bounded interval. So the IVT is not applicable to functions that are discontinuous somewhere in the interval. One possible example is $f(x) = 1/x$ because it is discontinuous at $x = 0$. Indeed, notice that while $f(-1) = -1$ and $f(1) = 1$, the value of $f$ is never 0 for $-1 \le x \le 1$.

2. (10 pts) Aircraft require longer takeoff distances, called takeoff rolls, at high altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

| Elevation (ft) | Sea level | 1000 | 2000 | 3000 | 4000 |
|---|---|---|---|---|---|
| Takeoff roll (ft) | 670 | 734 | 805 | 882 | 967 |

   Let $f(x)$ be such a function, where $x$ is elevation in ft and $f(x)$ is the length of the takeoff roll in ft. Since $f$ is an exponential function, it must be of the form $f(x) = Ab^x$. Now

$$670 = f(0) = Ab^0 = A.$$

   So $f(x) = 670b^x$. We can estimate $b$ by using any one of the other data points. I will choose $(4000, 967)$:

$$967 = f(4000) = 670b^{4000} \implies b^{4000} = \frac{967}{670} \implies b = \sqrt[4000]{\frac{967}{670}}.$$

   So

$$f(x) = 670 \sqrt[4000]{\frac{967}{670}}^{\,x}.$$

   I will check my work by verifying that my function fits one of the other data points, say $(2000, 805)$:

$$f(2000) = 670 \sqrt[4000]{\frac{967}{670}}^{\,2000} = 670\sqrt{\frac{967}{670}} \approx 804.9$$

   which is in good agreement with the table.

3. (a) (4 pts) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be an odd function and give an example of an odd function.

   A function $f(x)$ is odd if $f(-x) = -f(x)$ for every real number $x$.
   An example of such a function is $f(x) = \sin(x)$ as it is indeed true that $\sin(-x) = -\sin(x)$.

   (b) (6 pts) Is the following statement true or false? If $f$ and $g$ are odd functions then $f(g(x))$ is an odd function. If you think this is true, give a proof. If you think it is false, find a counterexample.

The statement is true. Let $f$ and $g$ be odd functions. Then

$$f(g(-x)) = f(-g(x)) = -f(g(x))$$

for all $x \in \mathbb{R}$. Hence $f(g(x))$ is an odd function.

4. (10 pts) Evaluate the following limit. Be sure to justify your work by referring to appropriate properties of limits we stated (and in some cases proved) in class.

$$\lim_{x \to -2} \ln\left(\frac{3x^2 - 2}{1 - 2x}\right)$$

We will use the following properties:

$$\lim_{x \to c} k = k \tag{1}$$

$$\lim_{x \to c} x = c \tag{2}$$

$$\lim_{x \to c} (bf(x)) = b \lim_{x \to c} x \qquad \text{if } \lim_{x \to c} f(x) \text{ exists} \tag{3}$$

$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \qquad \text{if } \lim_{x \to c} f(x) \text{ and } \lim_{x \to c} g(x) \text{ exist} \tag{4}$$

$$\lim_{x \to c} (f(x)g(x)) = \big(\lim_{x \to c} f(x)\big)\big(\lim_{x \to c} g(x)\big) \qquad \text{if } \lim_{x \to c} f(x) \text{ and } \lim_{x \to c} g(x) \text{ exist} \tag{5}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \qquad \text{if both limits exist and } \lim_{x \to c} g(x) \neq 0 \tag{6}$$

$$\lim_{x \to c} f(x) = f(c) \qquad \text{if } f(x) \text{ is continuous at } x = c \tag{7}$$

Now

$$\begin{aligned}
\lim_{x \to -2} x &= -2 && \text{by property 2} \\
\lim_{x \to -2} x^2 &= \big(\lim_{x \to -2} x\big)^2 = (-2)^2 = 4 && \text{by property 5} \\
\lim_{x \to -2} 3x^2 &= 3 \lim_{x \to -2} x^2 = 3 \cdot 4 = 12 && \text{by property 3} \\
\lim_{x \to -2} -2 &= -2 && \text{by property 1} \\
\lim_{x \to -2} (3x^2 - 2) &= \lim_{x \to -2} 3x^2 + \lim_{x \to -2} -2 = 12 - 2 = 10 && \text{by property 4} \\
\lim_{x \to -2} (-2x) &= -2\big(\lim_{x \to -2} x\big) = -2(-2) = 4 && \text{by property 3} \\
\lim_{x \to -2} (1 - 2x) &= 1 + 4 = 5 && \text{by property 4} \\
\lim_{x \to -2} \frac{3x^2 - 2}{1 - 2x} &= \frac{\lim_{x \to -2} (3x^2 - 2)}{\lim_{x \to -2} (1 - 2x)} = \frac{10}{5} = 2 && \text{by property 6} \\
\lim_{x \to -2} \ln\left(\frac{3x^2 - 2}{1 - 2x}\right) &= \lim_{y \to 2} \ln(y) = \ln(2) && \text{by property 7}
\end{aligned}$$

Alternately, we could argue that $f(x) = x$ is continuous function at all real numbers $x$, hence $g(x) = 3x^2 - 2$ and $h(x) = 1 - 2x$ are also continuous at all real numbers by Theorem 1.3 in Section 1.8. By the same theorem, $g(x)/h(x)$ is continuous at $x = -2$ since $h(-2) \neq 0$. By Theorem 1.4, $\ln\big(g(x)/h(x)\big)$ is also continuous at $x = -2$, so

$$\lim_{x \to -2} \ln\left(\frac{3x^2 - 2}{1 - 2x}\right) = \ln\left(\frac{3(-2)^2 - 2}{1 - 2(-2)}\right) = \ln(2).$$

5. (10 pts) **Extra credit problem.** Let $f$ and $g$ be real valued functions. Use the formal definition of the limit to give a rigorous proof that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are both convergent, then

$$\lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

Let $L_1 = \lim_{x\to\infty} f(x)$ and $L_2 = \lim_{x\to\infty} g(x)$. Then

$$\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = L_1 + L_2.$$

We need to show that for any $\epsilon > 0$ there exists a number $N$ large enough that if $x > N$ then $|f(x) + g(x) - (L_1 + L_2)| < \epsilon$. Since $\lim_{x\to\infty} f(x) = L_1$, there is a number $N_1$ big enough that if $x > N_1$ then $|f(x) - L_1| < \epsilon/2$. Similarly, there is a number $N_2$ big enough that if $x > N_2$ then $|f(x) - L_2| < \epsilon/2$. Let $N = \max(N_1, N_2)$. If $x > N$, then

$$\begin{aligned} |f(x)+g(x)-(L_1+L_2)| &= |f(x)-L_1+g(x)-L_2| \\ &\le |f(x)-L_1| + |g(x)-L_2| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

by the Triangle Inequality, which says that $|a+b| \le |a| + |b|$ for any two numbers $a$ and $b$.