Math 142 Exam 2 Solutions

Mar 31, 2015

1. (5 pts each) Give an example of each of the following. Be sure to justify your examples.
   (a) A function f for which $f'(0) = 0$ but $f''(0) \neq 0$.

   One such function is $f(x) = x^2$. Then $f'(x) = 2x$, so $f'(0) = 0$, and $f''(x) = 2$, so $f''(0) = 2$.

   (b) An invertible function that is not differentiable at $x = 0$.

   Let $f(x) = \sqrt[3]{x}$. Then f is invertible as $g(x) = x^3$ satisfies $g(f(x)) = \sqrt[3]{x}^3 = x$ and $f(g(x)) = \sqrt[3]{x^3} = x$. The derivative of f at $x = 0$ is

$$\begin{aligned} \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} \\ &= \lim_{x \to 0} \frac{1}{x^{2/3}} \end{aligned}$$

   As $x \to 0$, $x^{2/3} \to 0$, so $\frac{1}{x^{2/3}}$ is divergent. Hence $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$.

2. (5 pts each) If $P(x) = (x - a)^2 Q(x)$ where $Q(x)$ is a polynomial and $Q(x) \neq 0$, we call $x = a$ a double zero of the polynomial $P(x)$.
   (a) If $x = a$ is a double zero of a polynomial $P(x)$, show that $P(a) = P'(a) = 0$.

   If $x = a$ is a double zero of $P(x)$, then $P(x) = (x - a)^2 Q(x)$ for some polynomial $Q(x)$. Hence

$$\begin{aligned} P(a) &= (a - a)^2 Q(a) = 0 \\ P'(x) &= 2(x - a)Q(x) + (x - a)^2 Q'(x) \\ P'(x) &= 2(a - a)Q(a) + (a - a)^2 Q'(a) = 0 \end{aligned}$$

   (b) If $P(x)$ is a polynomial and $P(a) = P'(a) = 0$, show that $x = a$ is a double zero of $P(x)$.

   Since $P(a) = 0$, $x - a$ must be a factor or $P(x)$. I.e. there exists a polynomial $Q(x)$ such that $P(x) = (x - a)Q(x)$. Now

$$P'(x) = Q(x) + (x - a)Q'(x) \implies 0 = P'(a) = Q(a) + (a - a)Q'(a) = Q(a).$$

   Since $Q(a) = 0$, $x - a$ must be a factor of $Q(x)$. I.e. $Q(x) = (x - a)R(x)$ for some polynomial $R(x)$. Hence $P(x) = (x - a)^2 R(x)$. This shows $x = a$ is a double zero of $P(x)$.

3. (5 pts each)
   (a) Let

$$f(x) = \frac{7^{\sqrt[3]{x}}}{\ln(x) + 1}.$$

   Find $f'(1)$.

$$f'(x) = \frac{\left(\frac{d}{dx} 7^{\sqrt[3]{x}}\right)(\ln(x)+1) - 7^{\sqrt[3]{x}} \frac{d}{dx}(\ln(x)+1)}{(\ln(x)+1)^2}$$
$$= \frac{7^{\sqrt[3]{x}} \ln(7) \left(\frac{d}{dx}\sqrt[3]{x}\right)(\ln(x)+1) - 7^{\sqrt[3]{x}}\frac{1}{x}}{(\ln(x)+1)^2}$$
$$= \frac{7^{\sqrt[3]{x}} \ln(7) \frac{1}{3} x^{-2/3}(\ln(x)+1) - 7^{\sqrt[3]{x}}\frac{1}{x}}{(\ln(x)+1)^2}$$

Hence

$$f'(1) = \frac{7^{\sqrt[3]{1}} \ln(7) \frac{1}{3} 1^{-2/3}(\ln(1)+1) - 7^{\sqrt[3]{1}}\frac{1}{1}}{(\ln(1)+1)^2}$$
$$= \frac{7\ln(7)\frac{1}{3}(0+1) - 7}{(0+1)^2}$$
$$= \frac{7\ln(7)}{3} - 7$$

(b) Use the fact that $\tan(\arctan(x)) = x$ to find the derivative of $\arctan(x)$.

This is done in Section 3.6 in your textbook. See p. 157.

4. (10 pts) Let n be a positive integer. Use the definition of the derivative to prove that

$$\frac{d}{dx}x^n = nx^{n-1}.$$

This proof is done in Section 3.1 in your textbook using the Binomial Theorem. See p. 127.

Here is the proof we gave in class without the Binomial Theorem. Let $f(x) = x^n$ where n is positive integer.

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{x^n - x_0^n}{x - x_0}.$$

Use the identity

$$(x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}) = x^n - y^n$$

in the numerator:

$$\lim_{x\to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x\to x_0} \frac{(x-x_0)(x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-1})}{x - x_0}$$
$$= \lim_{x\to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-1})$$
$$= x_0^{n-1} + x_0^{n-2}x_0 + \cdots + x_0^{n-1}$$
$$= nx_0^{n-1}$$

Hence $\frac{d}{dx}x^n = nx^{n-1}$.

5. (5 pts each)**Extra credit problem.** We said that a function $f(x)$ is differentiable at x if

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. We say that f is *symmetrically differentiable* at x if

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

exists.

(a) We showed in class that $f(x)=|x|$ is not differentiable at $x=0$. Prove that it is symmetrically differentiable at $x=0$.

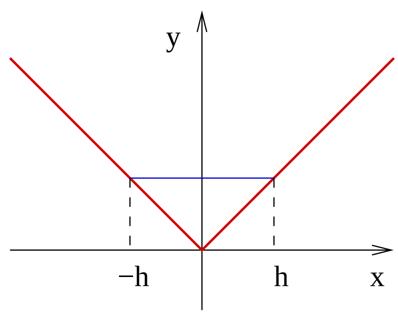
$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0-h)}{2h} &= \lim_{h\to 0}\frac{|h|-|-h|}{2h}\\ &= \lim_{h\to 0}\frac{0}{2h}=0\end{aligned}$$

Since this limit exists, $f(x)=|x|$ is symmetrically differentiable at $x=0$. Its symmetric derivative is 0 there.

Here is the geometric interpretation of what happens here. Notice that

$$\frac{f(0+h)-f(0-h)}{2h}$$

is the slope of the secant line between $x=-h$ and $x=h$. Because of the mirror symmetry of the graph about the y-axis, this secant line is horizontal for any value of h. This is why the symmetric derivative of $f(x)=|x|$ at $x=0$ exists and is 0.



(b) Another function that is not differentiable at $x=0$ is $f(x)=\sqrt[3]{x}$. Is it symmetrically differentiable at $x=0$?

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0-h)}{2h} &= \lim_{h\to 0}\frac{\sqrt[3]{h}-\sqrt[3]{-h}}{2h}\\ &= \lim_{h\to 0}\frac{\sqrt[3]{h}-(-\sqrt[3]{h})}{2h}\\ &= \lim_{h\to 0}\frac{2\sqrt[3]{h}}{2h}\\ &= \lim_{h\to 0}\frac{\sqrt[3]{h}}{h}\\ &= \lim_{h\to 0}\frac{1}{h^{2/3}}\end{aligned}$$

As $h\to 0$, $h^{2/3}\to 0$, so $\frac{1}{h^{2/3}}$ is divergent. Hence $f(x)=\sqrt[3]{x}$ is not symmetrically differentiable at $x=0$.

Notice that the secant line between $x = -h$ and $x = h$ gets steeper for smaller values of h. In the limit as $h \to 0$, the line is vertical, hence its slope is undefined.