Math 142 Final Exam Solutions

May 6, 2015

1. (10 pts) Using the equation of the tangent line to the graph of e^x at $x = 0$, show that
$$e^x \geq 1 + x$$
for all values of x. A sketch may be helpful. (Hint: Consider the concavity of e^x.)

To find the equation of the tangent line at $x = 0$, we need the slope
$$\frac{df}{dx} = e^x \implies f'(0) = e^0 = 1.$$
The tangent line must also pass through the point $(0, f(0)) = (0, 1)$. So its equation is $y = (x - 0) + 1 = x + 1$. Now notice that
$$\frac{d^2f}{dx^2} = e^x > 0$$
for all real numbers x. Hence the graph of $f(x)$ is concave up everywhere. Therefore it must be above the tangent line at all values of x. We can now conclude
$$e^x \geq 1 + x$$
for all $x \in \mathbb{R}$.

2. (10 pts) Consider a function f and a point a. Suppose there is a number L such that the linear function
$$g(x) = f(a) + L(x - a)$$
is a good approximation to f. By good approximation, we mean that
$$\lim_{x \to a} \frac{E_L(x)}{x - a} = 0$$
where $E_L(x)$ is the approximation error defined by
$$f(x) = g(x) + E_L(x) = f(a) + L(x - a) + E_L(x).$$
Show that f is differentiable at $x = a$ and that $f'(a) = L$. Thus the tangent line approximation is the only good linear approximation.

We actually did this argument in class, so it should be in your notes. Or here it is.
To see if $f(x)$ is differentiable at $x = a$, we look at the limit of the difference quotient
$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(a) + L(x - a) + E_L(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{L(x - a) + E_L(x)}{x - a} \\
&= \lim_{x \to a} \left(\frac{L(x - a)}{x - a} + \frac{E_L(x)}{x - a} \right) \\
&= \lim_{x \to a} \left(L + \frac{E_L(x)}{x - a} \right) \\
&= \lim_{x \to a} L + \lim_{x \to a} \frac{E_L(x)}{x - a} && \text{since both these limits exist} \\
&= L + 0 = L
\end{aligned}$$

This shows $f'(a)$ exists, hence f is differentiable at $x = a$, and that $f'(a) = L$ as desired.

3. (5 pts each) In this problem we prove a special case of the Mean Value Theorem where $f(a) = f(b) = 0$. This special case is called Rolle's Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b) = 0$, then there is a number c, with $a < c < b$, such that $f'(c) = 0$.

   By the Extreme Value Theorem, f has a global maximum and a global minimum on $[a, b]$.

   (a) Prove Rolle's Theorem in the case that both the global maximum and the global minimum are at endpoints of $[a, b]$. (Hint: $f(x)$ must be a very simple function in this case.)

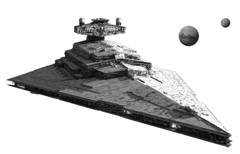
   If both the global minimum and the global maximum are at the endpoints of $[a, b]$ then they must both be equal to 0. Hence $f(x) = 0$ for all $x \in [a, b]$. Then $f'(x) = 0$ for all interior points $x \in (a, b)$. So any $c \in (a, b)$ will give $f'(c)$.

   (b) Prove Rolle's Theorem in the case that either the global maximum or the global minimum is not at an endpoint. (Hint: Think about local maxima and minima.)

   This means that either the global minimum or the global maximum is at an interior point $c \in [a, b]$. Any point where f has global minimum or maximum is also a local minimum or maximum. Since c is an interior point, it is in (a, b), and it must be a critical point of f by Theorem 4.1. This means either $f'(c) = 0$ or $f'(c)$ does not exist. Since f is differentiable everywhere on (a, b), $f'(c)$ must exist and hence it must be 0.

4. (10 pts) Prove that a function $f(x)$ that is differentiable at $x = a$ must be continuous at $x = a$.

   See the proof of Theorem 2.1 in your textbook.



5. Luke Skywalker and a band of Rebel Alliance fighters are hiding out from the Galactic Empire on the distant planet Hoth when rebel spies report that Darth Vader's star destroyer is approaching. According to the rebels' computations, the coordinates of Vader's ship at time t are given by $x = \sqrt{2t/e}$, $y = e^{-t}$, where x and y are measured in lightyears and t is measured in days starting right now. Hoth is at the point $(0, 0)$. The reconnaissance equipment on Vader's ship is able to detect the presence of lifeforms within 0.8 lightyears.

   (a) (6 pts) What is the closest distance Vader's ship passes to Hoth? (Hint: Can you minimize the square of the distance from a point on the curve to the origin instead of the distance? Why?)

   When the destroyer is at the point (x, y), its distance from the origin is $\sqrt{x^2 + y^2}$ by the Pythagorean Theorem.Since the square root function is an increasing function, $\sqrt{x^2 + y^2}$ has a minimum exactly where $x^2 + y^2$ has a minimum. That is the smallest possible (nonnegative) number under the square root will give the smallest possible value of the square root. To find the critical points of $x^2 + y^2$, we find the derivative

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}\left(\sqrt{\frac{2t}{e}}^2 + (e^{-t})^2\right) = \frac{d}{dt}\left(\frac{2t}{e} + e^{-2t}\right) = \frac{2}{e} + e^{-2t}(-2).$$

   Since e^{-2t} has a value for any $t \in \mathbb{R}$, the derivative exists for any $t \geq 0$ and it is 0 when

$$0 = \frac{2}{e} - 2e^{-2t} \implies \frac{2}{e} = 2e^{-2t} \implies \frac{1}{e} = e^{-2t} \implies -2t = -1 \implies t = \frac{1}{2}.$$

The value of x^2+y^2 at $t=1/2$ is

$$x^2+y^2|_{t=1/2} = \frac{2\frac{1}{2}}{e} + e^{-2\frac{1}{2}} = \frac{1}{e} + e^{-1} = \frac{2}{e}.$$

The value at the endpoint $t=0$ is

$$x^2+y^2|_{t=0} = \frac{0}{e} + e^0 = \frac{1}{e} + 1 > \frac{1}{e} + \frac{1}{e} = \frac{2}{e}.$$

Notice that

$$x^2+y^2 = \sqrt{\frac{2t}{e}}^2 + (e^{-t})^2 = \frac{2t}{e} + e^{-2t}$$

goes to ∞ as $t \to \infty$. Therefore the global minimum value of x^2+y^2 is at $t=1/2$, and it is $2/e$. This closest Vader's ship passes to Hoth is a distance of $\sqrt{2/e} \approx 0.858$ lightyears.

(b) (3 pts) How do you know that the value you found in part (a) is the global minimum of the distance?

Notice that

$$t > \frac{1}{2} \implies 2t > 1 \implies e^{-2t} < e^{-1} \implies \frac{2}{e} - 2e^{-2t} > 0.$$

That is x^2+y^2 is strictly increasing for $t > 1/2$. Similarly,

$$t < \frac{1}{2} \implies 2t < 1 \implies e^{-2t} > e^{-1} \implies \frac{2}{e} - 2e^{-2t} < 0.$$

That if x^2+y^2 is strictly decreasing for $t < 1/2$. Therefore the critical point at $t=1/2$ must be a both a local and a global minimum.

(c) (1 pt) Should the rebels worry about being detected?

No because they are never within 0.8 lightyears of Vader's ship.

6. (5 pts each)
(a) Evaluate

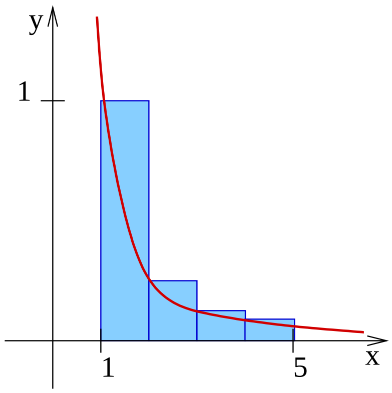
$$\int_1^5 \frac{1}{x^2}\,dx$$

Since $f(x) = 1/x^2$ is a continuous function between $x=1$ and $x=5$, we can use the Fundamental Theorem of Calculus to evaluate the integral. By writing $f(x) = x^{-2}$, we can tell that the antiderivatives of $f(x)$ are of the form $x^{-1}/(-1) + c = -1/x + c$. We only need one of these antiderivatives to use the FTC. I will choose $-1/x$. So

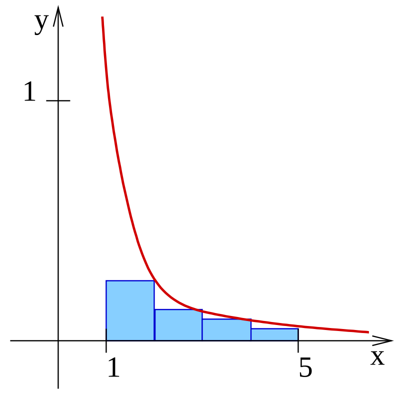
$$\int_1^5 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^5 = -\frac{1}{5} + \frac{1}{1} = \frac{4}{5}.$$

(b) Let I be the value of the integral in part (a). Let L and R be the values of the left-hand sum and the right-hand sum for the above integral with n subdivisions. Which has the largest value among I, L, and R? Which has the smallest value? Be sure to justify your answer.

The function $f(x) = 1/x^2$ is decreasing, so the rectangles used in the left-hand sum overestimate the area under the graph, while those used in the right-hand sum underestimate the area. You can see this in these diagrams:



LHS



RHS

Hence $L > I > R$.

7. (10 pts)**Extra credit problem.** Let $f(x)$ be a positive differentiable function such that $\lim\limits_{x\to\infty} f(x) = 0$. Does
$$\lim_{x\to\infty} f(x)^{f(x)}$$
exist? If so, find it. Otherwise explain why it doesn't exist. Remember that a complete answer includes justification. (Hint: Can you use L'Hopital's rule?)

You can do this problem like Example 8 in 4.7, or the two examples $\lim_{x\to\infty} \sqrt[x]{x}$ and $\lim_{x\to\infty}(1 + 1/x)^x$ we did in class.

First, rewrite
$$f(x)^{f(x)} = e^{\ln(f(x)^{f(x)})} = e^{f(x)\ln(f(x))}$$
Now as $x \to \infty$, $f(x) \to 0$, so $\ln(f(x)) \to -\infty$. Since $f(x) \neq 0$, we can write
$$f(x)\ln(f(x)) = \frac{\ln(f(x))}{\frac{1}{f(x)}}$$
Notice that $1/f(x) \to 0$ and
$$\frac{d}{dx}\frac{1}{f(x)} = \frac{d}{dx}[f(x)]^{-1} = -[f(x)]^2 f'(x) = -\frac{f'(x)}{[(f(x)]^2}$$
since $f(x)$ is differentiable and $[f(x)]^2 \neq 0$, this derivative always exists. That is $1/f(x)$ is differentiable. Therefore we can apply l'Hopital's Theorem to find
$$\begin{aligned}\lim_{x\to\infty} f(x)\ln(f(x)) &= \lim_{x\to\infty} \frac{\ln(f(x))}{\frac{1}{f(x)}} \\ &= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(f(x))}{\frac{d}{dx}\frac{1}{f(x)}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{f(x)}f'(x)}{-\frac{f'(x)}{[(f(x)]^2}}\end{aligned}$$
It is tempting to cancel $f'(x)$, but we need to be careful. What if it is 0? Since $f(x) \to 0$ while $f(x) > 0$, $f(x)$ cannot be a constant function. Therefore $f(x) \neq 0$ in general, although

there may be values of x for which $f'(x) = 0$. But we are looking at the limit as $x \to \infty$, so what happens at specific values of x is not really important. We are interested in the general trend of where these values are going as $x \to \infty$. Hence it is OK to cancel $f'(x)$ in the fraction above. We can also multiply both the numerator and the denominator by $[f(x)]^2$ as it is not 0. So

$$\lim_{x\to\infty} \frac{\frac{1}{f(x)} f'(x)}{-\frac{f'(x)}{[(f(x)]^2}} = \lim_{x\to\infty} \frac{\frac{1}{f(x)}}{-\frac{1}{[(f(x)]^2}} = \lim_{x\to\infty} -f(x) = 0$$

Now

$$x \to \infty \implies \ln\left(f(x)^{f(x)}\right) \to 0 \implies e^{\ln(f(x)^{f(x)})} \to e^0 = 1$$

since e^x is a continuous function. In conclusion,

$$\lim_{x\to\infty} f(x)^{f(x)} = 0.$$