Math 143 Final Exam Solutions

Dec 10, 2014

1. (10 pts) Let $F(a)$ be the area under the graph of $y = x^2e^x$ between $x = 0$ and $x = a$, for $a > 0$.

   (a) Find a formula for $F(a)$.

   Using integration by parts twice:

$$\begin{aligned}
F(a) &= \int_0^a \underbrace{x^2}_{g}\underbrace{e^x}_{f'}\,dx \\
&= x^2(e^x)\ |_0^a - \int_0^a 2x(e^x)dx \\
&= a^2e^a - 2xe^x\ |_0^a + \int_0^a 2e^x dx \\
&= a^2e^a - 2ae^a + 2e^x\ |_0^a \\
&= a^2e^a - 2ae^a + 2e^a - 2 \\
&= e^a(a^2 - 2a + 2) - 2
\end{aligned}$$

   (b) Is F an increasing or decreasing function?

   By the Fundamental Theorem of Calculus, $F'(a) = a^2e^a$. Notice that $a^2e^a \geq 0$ since $a^2 \geq 0$ and $e^a > 0$ for all a. So F is an increasing function.

   (c) Is F concave up or concave down for $0 < a < 2$?

$$F''(a) = \frac{d}{da}a^2e^a = 2ae^a + a^2e^a = e^a a(2 + a)$$

   Since $0 < a < 2$, $a > 0$ and $2 + a > 2$. As $e^a > 0$ for any a, $F''(a) > 0$. Hence F is concave up.

2. (10 pts)

   (a) Find an upper bound for

$$\int_3^\infty e^{-x^2} dx.$$

   (Hint: Show that $e^{-x^2} \leq e^{-3x}$ for $x \geq 3$ and then use this result.)

   First, notice that if $x \geq 3$ then $x^2 \geq 3x$. Hence $0 < e^{-x^2} \leq e^{-3x}$. Therefore

$$\begin{aligned}
0 < \int_3^\infty e^{-x^2} dx &\leq \int_3^\infty e^{-3x} dx \\
&= \lim_{a\to\infty} \int_3^a e^{-3x} dx \\
&= \lim_{a\to\infty} \left.\frac{e^{-3x}}{-3}\right|_3^a \\
&= \lim_{a\to\infty} \left(\frac{e^{-9}}{3} - \frac{e^{-3a}}{3}\right)
\end{aligned}$$

$$= \frac{e^{-9}}{3} - 0 = \frac{1}{3e^9}$$

Therefore both improper integrals are convergent and $1/(3e^9)$ is an upper bound for $\int_3^\infty e^{-x^2}\,dx$.

(b) Generalize the result of part (a) to find an upper bound for

$$\int_n^\infty e^{-x^2}\,dx$$

by noting (and proving) that $e^{-x^2} \le e^{-nx}$ for $x \ge n$.

This is very similar to part (a). First, notice that if $x \ge n$ then $x^2 \ge nx$. Hence $0 < e^{-x^2} \le e^{-nx}$. Therefore

$$\begin{aligned} 0 < \int_n^\infty e^{-x^2}\,dx &\le \int_n^\infty e^{-nx}\,dx \\ &= \lim_{a\to\infty} \int_n^a e^{-nx}\,dx \\ &= \lim_{a\to\infty} \left. \frac{e^{-nx}}{-n} \right|_n^a \\ &= \lim_{a\to\infty} \left(\frac{e^{-n^2}}{n} - \frac{e^{-na}}{n} \right) \\ &= \frac{e^{-n^2}}{n} - 0 = \frac{1}{ne^{n^2}} \end{aligned}$$

Therefore both improper integrals are convergent and $1/(ne^{n^2})$ is an upper bound for $\int_n^\infty e^{-x^2}\,dx$.

3. (10 pts) Decide if the following statement is true or false. If $\sum_{n=1}^\infty a_n b_n$ converges, then $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ converge. Justify your answer.

This is false. Here is a counterexample. Let $a_n = b_n = 1/n$. Then

$$\sum_{n=1}^\infty a_n b_n = \sum_{n=1}^\infty \frac{1}{n^2},$$

which is convergent because it is a p-series with $p > 1$. But

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n}$$

is divergent because it is the harmonic series.

4. (10 pts) Let S be the solid of revolution obtained by rotating the region bounded by the x-axis, $y = 1/(x+1)$, $x = 0$ and $x = 5$ about the x-axis. The density of S, measured in g/cm^3, varies along the x-axis as the function $\rho(x) = 1/(x+3)$ where x is in cm. Find the mass of S.

Divide the interval $[0, 5]$ into n parts of width Δx using the points $x_1 = 0, x_2, \ldots, x_{n+1} = 5$. Slice up the solid of revolution by planes through these points that are perpendicular to the x-axis. Each such slice is approximately a cylinder. In fact, the i-th slice is approximately a cylinder of radius $y_i = 1/(x_i + 1)$ and height Δx. Hence its volume is approximately $V_i \approx$

$\pi y_i^2 \Delta x$ and its mass is approximately $V_i \frac{1}{x_i+3}$. Hence the mass of the solid is approximated by the Riemann sum:

$$m \approx \sum_{i=0}^{n} \pi \frac{1}{(x_i+1)^2} \frac{1}{x_i+3} \Delta x.$$

The approximation becomes perfect as $n \to \infty$, so

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=0}^{n} \pi \frac{1}{(x_i+1)^2} \frac{1}{x_i+3} \Delta x \\ &= \int_0^5 \pi \frac{1}{(x+1)^2(x+3)} dx \end{aligned}$$

We will use partial fractions to evaluate this integral.

$$\begin{aligned} \frac{1}{(x+1)^2(x+3)} &= \frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{(x+3)} \\ &= \frac{A(x+3) + B(x+1)(x+3) + C(x+1)^2}{(x+1)^2(x+3)} \end{aligned}$$

Hence it must be true for all x that

$$1 = A(x+3) + B(x+1)(x+3) + C(x+1)^2.$$

Now, if $x = -1$, then

$$1 = A(-1+3) \implies A = \frac{1}{2}.$$

Now, if $x = -3$, then

$$1 = C(-3+1)^2 \implies C = \frac{1}{4}.$$

Finally, to find B, we set $x = 0$ and get

$$1 = 3A + 3B + C = \frac{3}{2} + 3B + \frac{1}{4} \implies B = -\frac{1}{4}.$$

Hence

$$\begin{aligned} \int_0^5 \pi \frac{1}{(x+1)^2(x+3)} dx &= \pi \int_0^5 \frac{1}{2(x+1)^2} - \frac{1}{4(x+1)} + \frac{1}{4(x+3)} dx \\ &= \pi \left[-\frac{1}{2(x+1)} - \frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x+3| \right]_0^5 \\ &= \pi \left(-\frac{1}{12} + \frac{1}{2} - \frac{1}{4}\big(\ln(6) - \ln(1)\big) + \frac{1}{4}\big(\ln(8) - \ln(3)\big) \right) \\ &= \frac{5\pi}{12} + \frac{\pi}{4} \ln\left(\frac{8}{6(3)}\right) \\ &= \frac{5\pi}{12} + \frac{\pi}{4} \ln\left(\frac{4}{9}\right) \\ &= \frac{5\pi}{12} + \frac{\pi}{2}(\ln(2) - \ln(3)) \end{aligned}$$

Therefore the mass of the solid is $5\pi/12 + \pi/2 \ln(2/3) \approx 0.6721$ grams.

5. (10 pts) After Rudolph chews up his Chinese internet stock certificates, Santa Claus sees no other option to fund his gift giving activities but to drill for oil under the Arctic ice. At time t years after opening the well, Santa extracts oil at a rate of $f(t) = 5\sin(\pi t/10)$ million barrels per year. He earns \$40 on each barrel of oil and immediately invests the money in a microbrewery which produces organic beer from Arctic moss and pays 20% annual interest on investments compounded continuously. How much money does Santa make in 10 years?

We can take the usual Riemann sum approach to this problem. Divide the interval $[0, 10]$ into n parts of width Δt at $t_1 = 0, t_2, \ldots, t_{n+1} = 10$. During the i-th interval, Santa extracts about $5\sin(\pi t_i/10)\Delta t$ million barrels of oil. So he earns about $200\sin(\pi t_i/10)\Delta t$ million dollars. He invests this money and earns interest for about $10 - t_i$ years, which makes him about $200\sin(\pi t_i/10)\Delta t e^{0.2(10-t_i)}$ million dollars. So his total income in 10 years is approximated by the Riemann sum

$$\sum_{i=1}^{n} 200\sin\left(\frac{\pi t}{10}\right)\Delta t e^{0.2(10-t_i)}.$$

The approximation becomes perfect as $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} 200\sin\left(\frac{\pi t}{10}\right)\Delta t e^{0.2(10-t_i)} &= \int_0^{10} 200\sin\left(\frac{\pi t}{10}\right) e^{0.2(10-t)}dt \\
&= 200e^2\int_0^{10} \sin\left(\frac{\pi t}{10}\right) e^{-0.2t}dt
\end{aligned}$$

To evaluate this integral, we will first find the indefinite integral $\int \sin(at)e^{bt}dt$ by doing integration by parts twice:

$$\begin{aligned}
\int \sin(at)e^{bt}dt &= \sin(at)\frac{e^{bt}}{b} - \int a\cos(at)\frac{e^{bt}}{b}dt \\
&= \sin(at)\frac{e^{bt}}{b} - a\cos(at)\frac{e^{bt}}{b^2} + \int a^2(-\sin(at))\frac{e^{bt}}{b^2}dt \\
&= \sin(at)\frac{e^{bt}}{b} - \frac{a}{b^2}\cos(at)e^{bt} - \frac{a^2}{b^2}\int \sin(at)e^{bt}dt
\end{aligned}$$

Now let $S = \int \sin(at)e^{bt}dt$. We get the equation

$$\begin{aligned}
S &= \sin(at)\frac{e^{bt}}{b} - \frac{a}{b^2}\cos(at)e^{bt} - \frac{a^2}{b^2}S \\
\left(1 + \frac{a^2}{b^2}\right)S &= \sin(at)\frac{e^{bt}}{b} - \frac{a}{b^2}\cos(at)e^{bt} \\
S &= \frac{b^2}{a^2+b^2}\left(\sin(at)\frac{e^{bt}}{b} - \frac{a}{b^2}\cos(at)e^{bt}\right) + c \\
&= \frac{e^{bt}}{a^2+b^2}(b\sin(at) - a\cos(at)) + c
\end{aligned}$$

Hence Santa's total income is

$$
\begin{aligned}
200e^2 \int_0^{10} \sin\left(\frac{\pi t}{10}\right) e^{-0.2t} dt \\
&= 200e^2 \frac{e^{-0.2t}}{(\pi/10)^2 + (-2/10)^2} \left(-\frac{2}{10}\sin\left(\frac{\pi t}{10}\right) - \frac{\pi}{10}\cos\left(\frac{\pi t}{10}\right)\right)\Bigg|_0^{10} \\
&= 200e^2 \left(\frac{100e^{-2}}{\pi^2+4}\left(-\frac{2}{10}\sin(\pi) - \frac{\pi}{10}\cos(\pi)\right) - \frac{100e^0}{\pi^2+4}\left(-\frac{2}{10}\sin(0) - \frac{\pi}{10}\cos(0)\right)\right) \\
&= 200e^2 \left(\frac{10\pi e^{-2}}{\pi^2+4} + \frac{10\pi}{\pi^2+4}\right) \\
&= \frac{2000\pi(1+e^2)}{\pi^2+4} \approx 3800.4
\end{aligned}
$$

So Santa makes about 3.8 billion dollars in 10 years. Now, is that enough to buy every child a gift in each of those years?

6. (5 pts each)
   (a) Evaluate

$$\int_1^\infty \frac{1}{x^2 - 6x + 25} dx$$

   if convergent. If it is divergent, explain why.

   We will work on the indefinite intergral first

$$
\begin{aligned}
\int \frac{1}{x^2-6x+25} dx &= \int \frac{1}{(x-3)^2+16} dx && \text{substitute } x-3 = 4\tan(y) \\
&= \int \frac{1}{(4\tan(y))^2+16} 4\sec^2(y)dy && \text{since } dx = 4\sec^2(y)dy \\
&= \int \frac{1}{16(\tan^2(y)+1)} 4\sec^2(y)dy \\
&= \int \frac{1}{16sec^2(y)} 4\sec^2(y)dy \\
&= \frac{1}{4}y + c \\
&= \frac{1}{4}\arctan\left(\frac{x-3}{4}\right) + c
\end{aligned}
$$

   Hence

$$
\begin{aligned}
\int_1^\infty \frac{1}{x^2-6x+25} dx &= \lim_{a\to\infty} \int_1^a \frac{1}{x^2-6x+25} dx \\
&= \lim_{a\to\infty} \frac{1}{4}\arctan\left(\frac{x-3}{4}\right)\Bigg|_1^a \\
&= \frac{1}{4}\left(\frac{\pi}{2} - \arctan\left(-\frac{1}{2}\right)\right)
\end{aligned}
$$

   Since this is a finite number, the integral is convergent.

   (b) Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2-6n+25}$ convergent or divergent? Carefully justify your answer.

Notice that the function $f(x) = \frac{1}{x^2-6x+25} = \frac{1}{(x-3)^2+16}$ is a positive decreasing function on $[1, \infty)$, whose improper integral on this interval is convergent by part (a). Hence by the Integral Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - 6n + 25}$ must be convergent.

7. (10 pts) **Extra credit problem.** Let $\sum_{n=1}^{\infty} a_n$ be a series. Prove that if $f(x)$ is a positive decreasing function such that $f(n) = a_n$ for all positive integers n and the improper integral

$$\int_1^{\infty} f(x)dx$$

is convergent then the series $\sum_{n=1}^{\infty} a_n$ must also be convergent.

Compare this problem with Example 4 in Section 9.3. The diagram in that example may help to follow the argument below.

Notice that $a_2 + a_3 + \ldots$ is a right-hand sum of the integral $\int_1^{\infty} f(x)dx$. Since f is decreasing, a right-hand sum is cannot be larger than the area under f. Hence

$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)dx.$$

Since the integral on the right is convergent, the series on the left is bounded from above. Since all of the a_i are positive, it is also an increasing series. But a bounded, increasing series must be convergent (Theorem 9.1). Therefore

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$$

is also convergent.