Math 143 Exam 1 Solutions

Sep 26, 2014

1. (10 pts) Explain in your own words how and why integration by substitution works. (Hint: you may want to use an example to illustrate your explanation, but an example by itself is not an explanation.)

   I will first explain why, then how it works. The chain rule says
   $$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$
   Taking the indefinite integral of both sides and using the definition of the indefinite integral, we get
   $$\int f'(g(x))g'(x)dx = \int \frac{d}{dx}f(g(x))dx = f(g(x)) + c$$
   where c is any constant. Now, given some integral with a composite function of the form $f'(g(x))g'(x)$, we can substitute $y = g(x)$ and $dy = \frac{dy}{dx}dx = g'(x)dx$ to get
   $$\int f'(g(x))g'(x)dx = \int f'(y)dy = f(y) + c = f(g(x)) + c.$$
   The advantage to doing this is that we can integrate the simpler function $f'(x)$ instead of the more complex original integrand.

   Even if the original integrand is not obviously in the form $f'(g(x))g'(x)$, we can substitute a new variable y for some part of the integrand, as long as we also substitute $dx = \frac{1}{(dy/dx)}dy$. This may or may not simplify the integration, but it is a valid thing to do because of the argument with the chain rule above. Since this works for indefinite integrals, it also works for definite integrals.

2. (10 pts) If f is a twice differentiable function, find
   $$\int f''(x)\ln(x)dx + \int \frac{f(x)}{x^2}dx.$$
   (Your answer should contain f, but no integrals.)

   By integration by parts,
   $$\int f''(x)\ln(x)dx = f'(x)\ln(x) - \int f'(x)\frac{1}{x}dx,$$
   and
   $$\int \frac{f(x)}{x^2}dx = f(x)\frac{-1}{x} - \int f'(x)\frac{-1}{x}dx = -f(x)\frac{1}{x} + \int f'(x)\frac{1}{x}dx.$$
   Add these together to get
   $$\int f''(x)\ln(x)dx + \int \frac{f(x)}{x^2}dx = f'(x)\ln(x) - \frac{f(x)}{x}.$$

3. (10 pts) Find the average of $\sin^2(x)$ between $x = -\pi/2$ and $x = \pi/2$.

   That average is
   $$\frac{1}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(x)dx$$

To evaluate the definite integral, I'll first find the indefinite integral $\int \sin^2(x)dx$ by parts:

$$\begin{aligned}
\int \sin^2(x)dx &= \int \underbrace{\sin(x)}_{f'}\underbrace{\sin(x)}_{g}\,dx \\
&= \underbrace{-\cos(x)}_{f}\underbrace{\sin(x)}_{g} - \int \underbrace{-\cos(x)}_{f}\underbrace{\cos(x)}_{g'}\,dx \\
&= -\sin(x)\cos(x) + \int \underbrace{\cos^2(x)}_{1-\sin^2(x)}\,dx \\
&= -\sin(x)\cos(x) + \int 1dx - \int \sin^2(x)dx \\
&= -\sin(x)\cos(x) + x - \int \sin^2(x)dx.
\end{aligned}$$

Now, if $A = \int \sin^2(x)dx$, then we have

$$\begin{aligned}
A = -\sin(x)\cos(x) + x - A &\implies 2A = -\sin(x)\cos(x) + x \\
&\implies A = \frac{-\sin(x)\cos(x) + x}{2}.
\end{aligned}$$

Hence the average is

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(x)dx = \frac{1}{\pi}\left(\frac{-\sin(x)\cos(x)+x}{2}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}.$$

4. (10 pts) Calculate the integral

$$\int \frac{x^3 - x^2 - 5x + 3}{x^2 - x - 6}dx.$$

and check your answer.

Since the degree of the numerator is at least the degree of the denominator, we'll start by dividing the numerator by the denominator

$$\begin{array}{r|l}
 & \qquad\qquad\quad x \\
x^2 - x - 6 & \;\; x^3 - x^2 - 5x + 3 \\
 & -x^3 + x^2 + 6x \\
\hline
 & \qquad\qquad\quad x
\end{array}$$

Since $x^2 - x - 6 = (x-3)(x+2)$, we now deal with the remainder using partial fractions:

$$\begin{aligned}
\frac{x+3}{(x-3)(x+2)} &= \frac{A}{x-3} + \frac{B}{x+2} \\
&= \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}.
\end{aligned}$$

Since this must be true for all real numbers x, we must have

$$x + 3 = A(x+2) + B(x-3).$$

In particular

$$\begin{aligned}
x = 3 &\implies 6 = A(3+2) \implies A = \frac{6}{5}, \\
x = -2 &\implies 1 = B(-2-3) \implies B = -\frac{1}{5}.
\end{aligned}$$

So

$$\int \frac{x^3 - x^2 - 5x + 3}{x^2 - x - 6} dx = \int x + \frac{\frac{6}{5}}{x-3} - \frac{\frac{1}{5}}{x+2} dx$$
$$= \frac{x^2}{2} + \frac{6}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| + c.$$

To check the answer

$$\frac{d}{dx}\left(\frac{x^2}{2} + \frac{6}{5}\ln|x-3| - \frac{1}{5}\ln|x+2|\right) = x + \frac{6}{5(x-3} - \frac{1}{5(x+2)}$$
$$= \frac{5x(x-3)(x+2) + 6(x+2) - (x-3)}{5(x-3)(x+2)}$$
$$= \frac{5x^3 - 5x^2 - 30x + 6x + 12 - x + 3}{5(x-3)(x+2)}$$
$$= \frac{5x^3 - 5x^2 - 25x + 15}{5(x-3)(x+2)}$$
$$= \frac{x^3 - x^2 - 5x + 3}{(x-3)(x+2)} \qquad \surd$$

5. (10 pts) **Extra credit problem.** Find

$$\int \arcsin(x) dx$$

and check your answer. (Hint: start by integration by parts, then do substitution. It may help to recall that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.)

Integrating by parts, we get

$$\int \underbrace{1}_{f'} \cdot \underbrace{\arcsin(x)}_{g} dx = \underbrace{x}_{f} \underbrace{\arcsin(x)}_{g} - \int \underbrace{x}_{f} \underbrace{\frac{1}{\sqrt{1-x^2}}}_{g'} dx$$
$$= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx.$$

Now, we will substitute $y = 1 - x^2$, hence $dy = -2xdx$:

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{1}{\sqrt{y}} dy$$
$$= -\frac{1}{2}\int \frac{\sqrt{y}}{\frac{1}{2}} + c$$
$$= -\sqrt{1-x^2} + c.$$

Putting the pieces together

$$\int \arcsin(x) dx = x\arcsin(x) + \sqrt{1-x^2} + c.$$

To check this answer,

$$\frac{d}{dx}\left(x\arcsin(x) + \sqrt{1-x^2}\right) = \arcsin(x) + x\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}(-2x)$$
$$= \arcsin(x) \qquad \surd$$