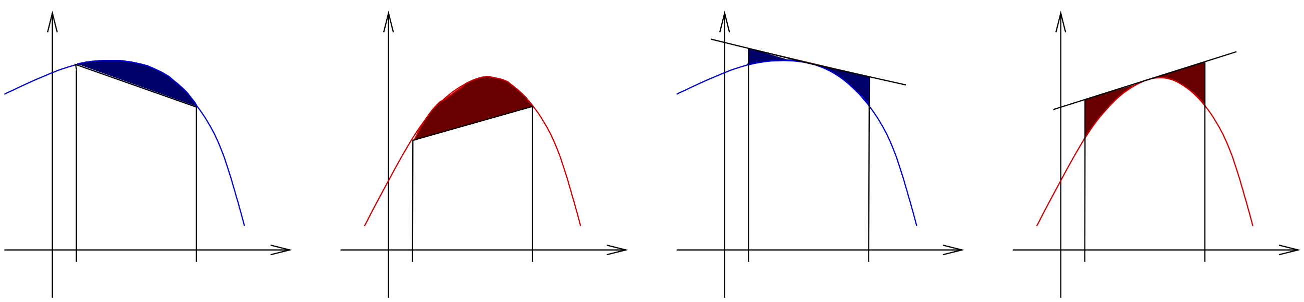
Math 143 Exam 2 Solutions

Oct 31, 2014

1. (10 pts) To investigate the relationship between the integrand and the errors in the midpoint and trapezoid rules, imagine an integrand whose graph is concave down over one subinterval of integration. Sketch graphs where f'' has small magnitude and where f'' has large magnitude. How do the errors compare?

   Both the midpoint and trapezoid rules use straight lines to approximate the function. Hence the error in both approximations is due to the function's curvature away from the straight line. The larger the magnitude of f'', the faster the slope of f is changing, i.e. the more the graph of f is curved. This is true whether f is concave up or down. Therefore when f'' has large magnitude, both the midpoint and the trapezoid rules will make a bigger error in approximating the area under the graph. You can see this on the diagrams below.



small f'', trapezoid rule   large f'', trapezoid rule   small f'', midpoint rule   large f'', midpoint rule

   Note: This is all I expected you to say. It is in fact possible to show that the for a function that is concave down (or up) throughout a subinterval, the midpoint rule gives a better approximation than the trapezoid rule. But the argument is somewhat subtle, although you already know all the math you need to give such argument.

   For a function f whose concavity is not uniform, you cannot say which of the two rules has smaller error, without knowing what f is. For many functions you would encounter as an example, the error of the trapezoid rule is about twice the error of the midpoint rule. This is because the typical function you think of is quite well approximated by a quadratic function on a short interval. But it is not hard to come up with a function for which the trapezoid rule gives a better approximation than the midpoint rule. E.g. for $\int_{-3\pi/2}^{3\pi/2} \cos(x)dx = -2$, $TRAP(1) = 0$ and $MID(1) = 3\pi$.

2. (10 pts) Decide if the following statement is true or false and justify your answer. If $f(x)$ is a positive periodic function, then $\int_0^\infty f(x)dx$ diverges.

   This is certainly true. Let P be the period of f. Let $A = \int_0^P f(x)dx$. Since $f(x) > 0$, $A > 0$. Now, the area under f from 0 to ∞ includes the area A again and again as f keeps going through its period again and again. Therefore $\int_0^\infty f(x)dx$ is infinitely large.

3. (10 pts) Is the improper integral

$$\int_{-\pi/2}^{\pi/2} \tan^2(x)dx$$

   convergent or divergent? If it is convergent, find its value.

This is a doubly improper integral, so we need to split it at some point between $-\pi/2$ and $\pi/2$. I'll choose 0 for this point.

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2(x)dx &= \int_{-\frac{\pi}{2}}^{0} \tan^2(x)dx + \int_{0}^{\frac{\pi}{2}} \tan^2(x)dx \\
&= \lim_{a\to -\frac{\pi}{2}^+} \int_{a}^{0} \tan^2(x)dx + \lim_{a\to \frac{\pi}{2}^-} \int_{0}^{a} \tan^2(x)dx
\end{aligned}$$

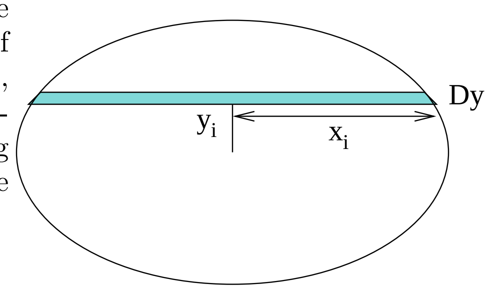
I will evaluate these one at a time.

$$\begin{aligned}
\lim_{a\to \frac{\pi}{2}^-} \int_{0}^{a} \tan^2(x)dx &= \lim_{a\to \frac{\pi}{2}^-} \int_{0}^{a} \sec^2(x) - 1dx \\
&= \lim_{a\to \frac{\pi}{2}^-} (\tan(x) - x)\ |_0^a \\
&= \lim_{a\to \frac{\pi}{2}^-} \tan(a) - a
\end{aligned}$$

As $a \to \pi/2^-$, $\tan(a) \to \infty$. Therefore this limit diverges to ∞. That is enough to make $\int_{-\pi/2}^{\pi/2} \tan^2(x)dx$ divergent. I do not even need to evaluate the other improper integral.

4. (10 pts) Use horizontal slices (strips) and a Riemann sum to find the area of an ellipse whose horizontal radius is a and vertical radius is b. (Hint: use either the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ or think of the ellipse as a circle that is stretched in the horizontal direction.)

I will compute the area of the upper half of the ellipse by dividing it up into horizontal slices. The width of the slice at y_i is $2x_i$. From the equation of the ellipse, $x_i = a\sqrt{1 - y^2/b^2}$. Each slice is approximately a rectangle, so its area is approximately $2x_i\Delta y$. Summing these and taking the limit as $n \to \infty$ gives me the exact area of the upper half of the ellipse:



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2a\sqrt{1 - \frac{y^2}{b^2}}\Delta y.$$

This is a Riemann sum, which is equal to the integral

$$\begin{aligned}
\int_0^b 2a\sqrt{1 - \frac{y^2}{b^2}}dy &= \frac{2a}{b}\int_0^b \sqrt{b^2 - y^2}dy && \text{substitute } y = b\sin(u) \\
&= \frac{2a}{b}\int_{y=0}^{y=b} \sqrt{b^2 - b^2\sin^{(} u)}\,b\cos(u)du && \text{since } dy = b\cos(u)du \\
&= 2a\int_{y=0}^{y=b} \sqrt{b^2\cos^2(u)}\cos(u)du \\
&= 2a\int_{y=0}^{y=b} b\cos^2(u)du \\
&= 2ab\int_{y=0}^{y=b} \cos^2(u)du
\end{aligned}$$

Now,

$$\begin{aligned}
\int \cos^2(u)du &= \int \underbrace{\cos(u)}_{f'}\underbrace{\cos(u)}_{g}\,du \\
&= \underbrace{\sin(u)}_{f}\underbrace{\cos(u)}_{g} - \int \underbrace{\sin(u)}_{f}\underbrace{(-\sin(u))}_{g'}\,du \\
&= \sin(u)\cos(u) + \int \underbrace{\sin^2(u)}_{1-\cos^2(u)}\,du \\
&= \sin(u)\cos(u) + \int 1du - \int \cos^2(u)du \\
&= \sin(u)\cos(u) + u - \int \cos^2(u)du.
\end{aligned}$$

Hence

$$\int \cos^2(u)du = \frac{1}{2}(\sin(u)\cos(u) + u) + c.$$

Now, back to the definite integral above

$$\begin{aligned}
2ab\int_{y=0}^{y=b} \cos^2(u)du &= 2ab\frac{\sin(u)\cos(u)+u}{2}\Bigg|_{y=0}^{y=b} && \text{substitute back } \sin(u) = y/b \\
&= 2ab\frac{\frac{y}{b}\sqrt{1-\frac{y^2}{b^2}} + \arcsin\left(\frac{y}{b}\right)}{2}\Bigg|_0^b && \text{as } \cos(u) = \sqrt{1-\sin^2(u)} = \sqrt{1-y^2/b^2} \\
&= ab\frac{\pi}{2}
\end{aligned}$$

This was the area of the upper half of the ellipse, so the area of the entire ellipse is πab.

5. (10 pts) **Extra credit problem.** Let $n \geq 2$ be an integer. Is the improper integral

$$\int_0^\infty \frac{1}{e^{\sqrt[n]{x}}}dx$$

convergent or divergent? Justify your answer.

Your intuition may tell you that this integral should converge. Even though $\sqrt[n]{x}$ may grow slowly as $x \to \infty$, the exponential function grows so fast that $e^{\sqrt[n]{x}}$ gets large quickly, and hence $1/e^{\sqrt[n]{x}} \to 0$ fast. To make this precise, we will do a comparison. We know $\int_1^\infty 1/x^2dx$ is convergent from class. Notice $1/x^2 = 1/e^{2\ln(x)}$. As x gets large, $\ln(x)$ will grow very slowly and eventually $\sqrt[n]{x}$ should outgrow $2\ln(x)$. So there should be some number a such that for $x > a$,

$$\sqrt[n]{x} > 2\ln(x) \implies e^{\sqrt[n]{x}} > e^{2\ln(x)} = x^2 \implies 1/e^{\sqrt[n]{x}} < 1/x^2$$

Hence

$$\int_a^\infty \frac{1}{e^{\sqrt[n]{x}}}dx \leq \int_a^\infty \frac{1}{x^2}dx$$

and the latter integral is convergent. So

$$\int_0^\infty \frac{1}{e^{\sqrt[n]{x}}}dx = \underbrace{\int_0^a \frac{1}{e^{\sqrt[n]{x}}}dx}_{\text{some finite number}} + \underbrace{\int_a^\infty \frac{1}{e^{\sqrt[n]{x}}}dx}_{\text{some finite number}}.$$

The only thing needed to make this argument precise is to find an estimate for a, so we can be certain that such a number indeed exists.

Notice that $e^x > x^2$ for any number $x \geq 2$. This is because $e^2 > 2^2$ and the exponential function grows faster than x^2 from this point on. We know $n > 1$, so $2n > 2$, and therefore $e^{2n} > (2n)^2$. Hence

$$\sqrt[n]{e^{2n^2}} = \sqrt[n]{(e^{2n})^n} = e^{2n} > (2n)^2 = 2(2n^2) = 2\ln(e^{2n^2}).$$

So let $a = e^{2n^2}$. Then $\sqrt[n]{a} > 2\ln(a)$. Since $2\ln(x)$ grows slower than $\sqrt[n]{x}$ (you can see this by comparing their derivatives) it is also true for any $x \geq a$ that $\sqrt[n]{x} > 2\ln(x)$. That is if $x \geq e^{2n^2}$, then

$$\sqrt[n]{x} > 2\ln(x) \implies e^{\sqrt[n]{x}} > e^{2\ln(x)} = x^2 \implies \frac{1}{e^{\sqrt[n]{x}}} < \frac{1}{x^2} \implies \int_{e^{2n^2}}^{\infty} \frac{1}{e^{\sqrt[n]{x}}} < \int_{e^{2n^2}}^{\infty} \frac{1}{x^2} < \infty.$$