Math 143 Final Exam Solutions

Dec 12, 2014

1. (10 pts) Evaluate the indefinite integral below, using a trigonometric substitution. Express the answer in terms of x.

$$\int \frac{1}{(25+4x^2)^{3/2}}dx.$$

I will substitute $x = 5/2\tan(y)$, so $4x^2 = 25\tan^2(y)$ and $dx = 5/2\sec^2(y)dy$.

$$\begin{aligned}
\int \frac{1}{(25+4x^2)^{3/2}}dx &= \int \frac{1}{(25+25\tan^2(y))^{3/2}}\frac{5}{2}\sec^2(y)dy \\
&= \int \frac{1}{(25\sec^2(y))^{3/2}}\frac{5}{2}\sec^2(y)dy \\
&= \int \frac{1}{125\sec^3(y)}\frac{5}{2}\sec^2(y)dy \\
&= \int \frac{1}{50\sec(y)}dy \\
&= \frac{1}{50}\int \cos(y)dy \\
&= \frac{1}{50}\sin(y) + c \qquad\qquad \text{subsitute } y = \arctan(2/5x) \\
&= \frac{1}{50}\sin\left(\arctan\left(\frac{2x}{5}\right)\right) + c
\end{aligned}$$

To simplify this, consider the right triangle to the right. In this triangle

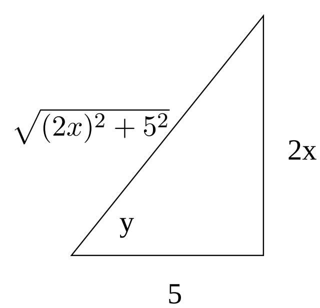
$$\tan(y) = \frac{2x}{5}$$

and

$$\sin(y) = \frac{2x}{\sqrt{4x^2+25}}.$$

So

$$\int \frac{1}{(25+4x^2)^{3/2}}dx = \frac{2x}{50\sqrt{4x^2+25}} + c.$$



2. (10 pts) Assuming $g(x)$ is a differentiable function whose values are bounded for all x, derive Stein's identity, which is used in statistics:

$$\int_{-\infty}^{\infty} g'(x)e^{-x^2/2}dx = \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}dx.$$

Since the integral is improper at both bounds, we need to split it up as the sum of two improper integrals.

$$\int_{-\infty}^{\infty} g'(x)e^{-x^2/2}dx = \lim_{a\to-\infty}\int_a^0 g'(x)e^{-x^2/2}dx + \lim_{a\to\infty}\int_0^a g'(x)e^{-x^2/2}dx$$

First, we will use integration by parts to work on the indefinite integral.

$$\begin{aligned}\int g'(x)e^{-x^2/2}dx &= g(x)e^{-x^2/2} - \int g(x)e^{-x^2/2}\frac{-2x}{2}dx \\ &= g(x)e^{-x^2/2} + \int xg(x)e^{-x^2/2}dx\end{aligned}$$

Now, we will use this result with the improper integrals, one at a time.

$$\begin{aligned}\lim_{a\to-\infty}\int_a^0 g'(x)e^{-x^2/2}dx &= \lim_{a\to-\infty}\left(g(x)e^{-x^2/2}\Big|_a^0 + \int_a^0 xg(x)e^{-x^2/2}dx\right) \\ &= \lim_{a\to-\infty}\left(g(0)e^0 - g(a)e^{-a^2/2} + \int_a^0 xg(x)e^{-x^2/2}dx\right) \\ &= g(0) - \underbrace{\lim_{a\to-\infty} g(a)e^{-a^2/2}}_{=0} + \int_{-\infty}^0 xg(x)e^{-x^2/2}dx \\ &= g(0) + \int_{-\infty}^0 xg(x)e^{-x^2/2}dx\end{aligned}$$

where

$$\lim_{a\to-\infty} g(a)e^{-a^2/2} = 0$$

because as $a \to \infty$, $-a^2/2 \to -\infty$ and $e^{-a^2/2} \to 0$, while $g(a)$ is bounded between two finite bounds, so $g(a)e^{-a^2/2} \to 0$.

Similarly,

$$\lim_{a\to-\infty}\int_a^0 g'(x)e^{-x^2/2}dx = -g(0) + \int_0^\infty xg(x)e^{-x^2/2}dx.$$

Hence

$$\begin{aligned}\int_{-\infty}^\infty g'(x)e^{-x^2/2}dx &= g(0) + \int_{-\infty}^0 xg(x)e^{-x^2/2}dx - g(0) + \int_0^\infty xg(x)e^{-x^2/2}dx \\ &= \int_{-\infty}^\infty xg(x)e^{-x^2/2}dx.\end{aligned}$$

3. (10 pts) Give an example of a convergent series $\sum_{n=1}^\infty a_n$ whose terms are all positive, such that the series $\sum_{n=1}^\infty \sqrt{a_n}$ is not convergent. Be sure to justify your example.

   Let $a_n = 1/n^2$. Then

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^2},$$

which is convergent because it is a p-series with $p > 1$. But

$$\sum_{n=1}^\infty \sqrt{a_n} = \sum_{n=1}^\infty \frac{1}{n}$$

is divergent because it is the harmonic series.

4. (10 pts) Let S be the solid of revolution obtained by rotating the region bounded by the x-axis, $y = \frac{1}{(x+2)\sqrt{x}}$, $x = 1$ and $x = 7$ about the x-axis. By symmetry, the center of mass of S must be on the x-axis. Find its location.

As usual, we can find the center of mass by dividing the moment by the total mass. To find the moment and the mass, we divide $[1,7]$ into n intervals of width Δx at $x_1 = 1, x_2, \ldots, x_{n+1} = 7$ and slice the solid into vertical slices by planes perpendicular to the x-axis at each point. Each such vertical slice is approximately a cylinder. The i-th slice has radius $y_i = \frac{1}{(x_i+2)\sqrt{x_i}}$ and height Δx. So its volume is approximately $V_i \approx \pi y_i^2 \Delta x$. If the solid has density ρ, then the mass of the i-th slice is $m_i = V_i \rho \approx \rho\pi y_i^2 \Delta x$ and the moment of that slice is approximately $M_i \approx x_i m_i \approx x_i \rho \pi y_i^2 \Delta x$. So the moment and the mass of the object are

$$M \approx \sum_{i=1}^{n} x_i \rho \pi y_i^2 \Delta x = \sum_{i=1}^{n} x_i \rho \pi \frac{1}{(x_i+2)^2 x_i} \Delta x$$
$$m \approx \sum_{i=1}^{n} \rho \pi y_i^2 \Delta x = \sum_{i=1}^{n} \rho \pi \frac{1}{(x_i+2)^2 x_i} \Delta x$$

In the limit, as $n \to \infty$, these approximations become precise and the Riemann sums become integrals:

$$\overline{x} = \frac{\int_1^7 \cancel{x}\cancel{\rho}\cancel{\pi} \frac{1}{(x+2)^2 \cancel{x}} dx}{\int_1^7 \cancel{\rho}\cancel{\pi} \frac{1}{(x+2)^2 x} dx} = \frac{\int_1^7 \frac{1}{(x+2)^2} dx}{\int_1^7 \frac{1}{(x+2)^2 x} dx}.$$

I will evaluate the definite integrals one at a time.

$$\int_1^7 \frac{1}{(x+2)^2} dx = \left.\frac{(x+2)^{-1}}{-1}\right|_1^7 = -\frac{1}{9} + \frac{1}{3} = \frac{2}{9}$$

For the other integral, we first need to rewrite the integrand in terms of partial fractions.

$$\begin{aligned}
\frac{1}{(x+2)^2 x} &= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x} \\
\frac{1}{(x+2)^2 x} &= \frac{A(x+2)x + Bx + C(x+2)^2}{(x+2)^2 x} \\
1 &= A(x+2)x + Bx + C(x+2)^2
\end{aligned}$$

$$\begin{aligned}
x = -2 &\implies 1 = B(-2) \implies B = -\frac{1}{2} \\
x = 0 &\implies 1 = C2^2 \implies C = \frac{1}{4} \\
x = -1 &\implies 1 = A(-1) + B(-1) + C1^2 = -A + \frac{1}{2} + \frac{1}{4} \implies A = -\frac{1}{4}
\end{aligned}$$

So

$$\begin{aligned}
\int_1^7 \frac{1}{(x+2)^2 x} dx &= \int_1^7 -\frac{1}{4(x+2)} - \frac{1}{2(x+2)^2} + \frac{1}{4x} dx \\
&= \left.\left(-\frac{1}{4}\ln|x+2| - \frac{1}{2}\frac{(x+2)^{-1}}{-1} + \frac{1}{4}\ln|x|\right)\right|_1^7 \\
&= -\frac{1}{4}\big(\underbrace{\ln(9) - \ln(3)}_{2\ln(3)-\ln(3)}\big) + \frac{1}{2}\left(\frac{1}{9} - \frac{1}{3}\right) + \frac{1}{4}\big(\ln(7) - \ln(1)\big) \\
&= -\frac{1}{4}\ln(3) - \frac{1}{9} + \frac{1}{4}\ln(7)
\end{aligned}$$

So the center of mass is at the point

$$\overline{x} = \frac{\frac{2}{9}}{\frac{1}{4}\ln\left(\frac{7}{3}\right)) - \frac{1}{9}} = \frac{8}{9\ln(7/3) - 4} \approx 2.21.$$

5. (10 pts) After Rudolph chews up his Chinese internet stock certificates, Santa Claus sees no other option to fund his gift giving activities but to drill for oil under the Arctic ice. At time t years after opening the well, Santa extracts oil at a rate of $f(t) = 3\cos(\pi t/30)$ million barrels per year until the well runs dry. Each barrel of oil is sold at a profit of \$40. Suppose Santa could invest money at 10% annual interest compounded continuously. What is the present value of all his revenue from oil extraction?

First, notice that the well runs dry when the rate of extraction slows to 0. That is when

$$3\cos\left(\frac{\pi t}{30}\right) = 0$$
$$\cos\left(\frac{\pi t}{30}\right) = 0$$
$$\frac{\pi t}{30} = \frac{\pi}{2} + k\pi$$

Of all these solutions, we want the first positive one. I.e.

$$\frac{\pi t}{30} = \frac{\pi}{2} \implies t = \frac{30}{2} = 15.$$

We can now take the usual Riemann sum approach to this problem. Divide the interval $[0, 15]$ into n parts of width Δt at $t_1 = 0, t_2, \ldots, t_{n+1} = 15$. During the i-th interval, Santa extracts about $3\cos(\pi t_i/30)\Delta t$ million barrels of oil. So he earns about $120\cos(\pi t_i/30)\Delta t$ million dollars. The present value of this money is about $120\cos(\pi t_i/30)e^{-0.1t_i}$. So the present value of all his earnings is approximated by the Riemann sum

$$\sum_{i=1}^{n} 120\cos\left(\frac{\pi t}{30}\right)\Delta t e^{-0.1t_i}.$$

The approximation becomes perfect as $n \to \infty$:

$$\lim_{n\to\infty}\sum_{i=1}^{n} 120\cos\left(\frac{\pi t}{30}\right)\Delta t e^{-0.1t_i} = \int_0^{15} 120\cos\left(\frac{\pi t}{30}\right)e^{-0.1t}dt$$

To evaluate this integral, we will first find the indefinite integral $\int \cos(at)e^{bt}dt$ by doing integration by parts twice:

$$\begin{aligned}\int \cos(at)e^{bt}dt &= \cos(at)\frac{e^{bt}}{b} - \int a(-\sin(at))\frac{e^{bt}}{b}dt \\ &= \cos(at)\frac{e^{bt}}{b} + a\sin(at)\frac{e^{bt}}{b^2} - \int a^2\cos(at)\frac{e^{bt}}{b^2}dt \\ &= \cos(at)\frac{e^{bt}}{b} + \frac{a}{b^2}\sin(at)e^{bt} - \frac{a^2}{b^2}\int \cos(at)e^{bt}dt\end{aligned}$$

Now let $S = \int \cos(at)e^{bt}dt$. We get the equation

$$\begin{aligned} S &= \cos(at)\frac{e^{bt}}{b} + \frac{a}{b^2}\sin(at)e^{bt} - \frac{a^2}{b^2}S \\ \left(1+\frac{a^2}{b^2}\right)S &= \cos(at)\frac{e^{bt}}{b} + \frac{a}{b^2}\sin(at)e^{bt} \\ S &= \frac{b^2}{a^2+b^2}\left(\cos(at)\frac{e^{bt}}{b} + \frac{a}{b^2}\sin(at)e^{bt}\right) + c \\ &= \frac{e^{bt}}{a^2+b^2}(b\cos(at) + a\sin(at)) + c \end{aligned}$$

Hence the present value of Santa's oil revenue is

$$\begin{aligned} &\int_0^{15} 120\cos\left(\frac{\pi t}{30}\right)e^{-0.1t}dt \\ &= 120\,\frac{e^{-0.1t}}{\left(\frac{\pi}{30}\right)^2 + \left(-\frac{1}{10}\right)^2}\left(-\frac{1}{10}\cos\left(\frac{\pi t}{30}\right) + \frac{\pi}{30}\sin\left(\frac{\pi t}{30}\right)\right)\Bigg|_0^{15} \\ &= 120\left(\frac{900e^{-1.5}}{\pi^2+9}\left(-\frac{1}{10}\cos\left(\frac{\pi}{2}\right) + \frac{\pi}{30}\sin\left(\frac{\pi}{2}\right)\right) - \frac{900e^0}{\pi^2+9}\left(-\frac{1}{10}\cos(0) + \frac{\pi}{30}\sin(0)\right)\right) \\ &= 120\left(\frac{900}{e^{1.5}(\pi^2+9)}\frac{\pi}{30} - \frac{900}{\pi^2+9}\left(-\frac{1}{10}\right)\right) \\ &= 120\left(\frac{30\pi}{e^{1.5}(\pi^2+9)} + \frac{90}{\pi^2+9}\right) \\ &\approx 706.1 \text{ million dollars} \end{aligned}$$

6. (5 pts each) Are the infinite series below convergent or divergent? Carefully justify your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

   This is the harmonic series. It is a p-series with $p = 1$. We showed in class that p-series converge if and only if $p > 1$ by using the Integral Test. So this series is divergent.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   This series is also divergent because it is a p-series with $p = 1/2$.

7. (10 pts) **Extra credit problem.** Let $\sum_{n=1}^{\infty} a_n$ be a series. Prove that if $f(x)$ is a positive decreasing function such that $f(n) = a_n$ for all positive integers n and the improper integral

$$\int_1^{\infty} f(x)dx$$

is convergent then the series $\sum_{n=1}^{\infty} a_n$ must also be convergent.

Compare this problem with Example 4 in Section 9.3. The diagram in that example may help to follow the argument below.

Notice that $a_2 + a_3 + \ldots$ is a right-hand sum of the integral $\int_1^\infty f(x)dx$. Since f is decreasing, a right-hand sum is cannot be larger than the area under f. Hence

$$\sum_{n=2}^{\infty} a_n \leq \int_1^\infty f(x)dx.$$

Since the integral on the right is convergent, the series on the left is bounded from above. Since all of the a_i are positive, it is also an increasing series. But a bounded, increasing series must be convergent (Theorem 9.1). Therefore

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$$

is also convergent.