Math 15 Midterm Solutions

S'01, May 7, 2001

1. (5 pts) Find $f^{-1}(3)$, if

$$f(x) = \frac{x+5}{x+1}$$

$$\begin{aligned} \frac{x+5}{x+1} &= 3 \\ x+5 &= 3x+3 \\ 2 &= 2x \\ x &= 1 \end{aligned}$$

So $f^{-1}(3) = 1$.

2. (5 pts) Find an equation of the line through $(3, 4)$ and $(-1, -2)$.

$$\text{slope} = \frac{4-(-2)}{3-(-1)} = \frac{6}{4} = \frac{3}{2}$$

So the equation is

$$\begin{aligned} \frac{y-(-2)}{x-(-1)} &= \frac{3}{2} \\ y+2 &= \frac{3}{2}(x+1) \\ y &= \frac{3}{2}x + \frac{3}{2} - 2 \\ y &= \frac{3}{2}x - \frac{1}{2} \end{aligned}$$

3. (5 pts each)

(a) Show that

$$2 - 3\log 5 = \log\left(\frac{4}{5}\right)$$

Notice $2 = \log 100$, so

$$2 - 3\log 5 = \log 100 - \log 5^3 = \log\left(\frac{100}{125}\right) = \log\left(\frac{4}{5}\right).$$

(b) Let $a = \log x$ and $b = \log y$. Express in terms of a and b

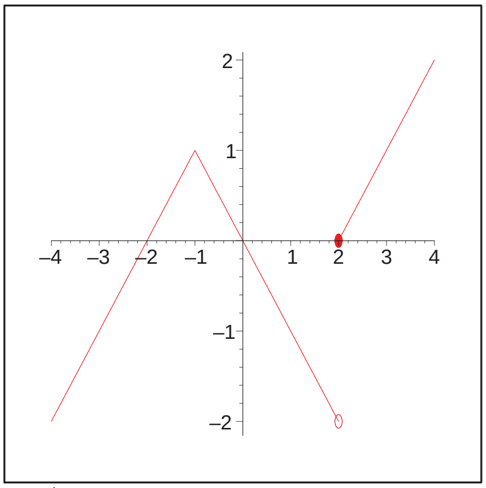
$$\log\left(\frac{\sqrt[3]{x}}{\sqrt{y^7}}\right).$$

$$\log\left(\frac{\sqrt[3]{x}}{\sqrt{y^7}}\right) = \log\left(\frac{x^{1/3}}{(y^7)^{1/2}}\right) = \log x^{1/3} - \log y^{7/2} = \frac{\log x}{3} - \frac{7}{2}\log y = \frac{a}{3} - \frac{7b}{2}$$

4. (10 pts) Graph the function

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ -x & \text{if } -1 \le x < 2 \\ x-2 & \text{if } 2 \le x \end{cases}.$$

Find $f(f(3))$.



$$f(f(3)) = f(3-2) = f(1) = -1$$

5. (10 pts) Find the domain and the range of
$$f(x) = \frac{1}{\sqrt{x-1}}.$$
For the domain, you want $x-1 \geq 0$ because of the square root and $x-1 \neq 0$ because of the division. So $x-1 > 0$, that is $x > 1$ and $\mathrm{D}(f) = \{x \mid x > 1\}$.

For the range, notice that $x-1$ can be any positive number, so $\sqrt{x-1}$ can be any positive number, so $\frac{1}{\sqrt{x-1}}$ can be any positive number too. Alternately:
$$\begin{aligned}
\frac{1}{\sqrt{x-1}} &= y \qquad \text{hence } y \geq 0 \\
\frac{1}{x-1} &= y^2 \\
1 &= (x-1)y^2 \qquad \text{shows } y \neq 0 \\
\frac{1}{y^2} &= x-1 \\
\frac{1}{y^2}+1 &= x
\end{aligned}$$
which gives some x for any $y \neq 0$. But we already know $y \geq 0$. Hence $\mathrm{R}(f) = \{y \mid y > 0\}$.

6. (15 pts) A bacterium culture is growing exponentially. After 1 day, it weighs 60 g, after 3 days 135 g.
   (a) Find a function $f(t)$ that describes the weight of the culture after t days.
   We know $f(t) = ab^t$ for some a and b. Now
$$\begin{aligned}
(1) \quad 60 &= ab \\
(2) \quad 135 &= ab^3
\end{aligned}$$
   Divide (2) by (1) to get
$$\frac{135}{60} = \frac{ab^3}{ab} \implies \frac{9}{4} = b^2 \implies b = \frac{3}{2}$$
   Now use (1) to find a:
$$\begin{aligned}
60 &= a\frac{3}{2} \\
40 &= a
\end{aligned}$$
   So $f(t) = 40\left(\frac{3}{2}\right)^t$.
   (b) What was the weight of the bacteria initially?
   $f(0) = 40$ so there were 40 g of bacteria initially.
   (c) What is the percent increase per day?
   $\frac{3}{2} = 150\%$, so the increase is 50%.
   (d) Assuming the growth were linear at a rate of 50%/day, how much would the culture weigh after 3 days?
   So it would grow by $50\%(40\,\mathrm{g}) = 20\,\mathrm{g}$ on the first day, and the 20 g each day to 100 g after 3 days.