Math 2111 Written Assignment 1
due 9/12/05

1. Let $f(x) = x^3 - 3x^2 + 5x - 3$
   (a) Find all roots of $f(x)$ over the complex numbers. (Hint: there is an obvious root. If you can't guess it, graph the polynomial.)
   (b) How many roots does $f(x)$ have? How many of them are real?
   (c) Do you notice anything special about the two complex roots?
2. (a) Let $f(x)$ be a polynomial with real coefficients. Suppose that $\alpha \in \mathbb{C}$ is such that $f(\alpha) = 0$. Prove that $f(\overline{\alpha}) = 0$. (Hint: Show first that if α and β are arbitrary complex numbers, $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$ and $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$. Now use these to argue $f(\overline{\alpha}) = \overline{f(\alpha)}$.)
   (b) Use the previous result to show that the complex roots of a real polynomial come in pairs of conjugates.
   (c) Let $f(x)$ be a real polynomial of odd degree. Prove that $f(x)$ has a real root.
3. In this exercise, you will give a slick (much more elegant than in class) proof of the sine and cosine addition formulas. Let $a = e^{\alpha i}$ and $b = e^{\beta i}$.
   (a) Find ab and write it in Cartesian coordinates.
   (b) Write a and b in Cartesian coordinates and then multiply them.
   (c) Compare the previous two results and derive the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.
4. Let p be a prime. It can be proven that in $\mathbb{Z}_p$, there is always an element $\overline{k}$ such that every nonzero element is equal to some power of $\overline{k}$. Luckily, you don't have to prove this now. Find an example of such a $\overline{k}$ in each of the sets below and show how every nonzero element arises as a power of $\overline{k}$.
   (a) $\mathbb{Z}_5$
   (b) $\mathbb{Z}_7$
   (c) $\mathbb{Z}_{11}$
5. Let $f(x) = x^2 + x - 2$.
   (a) Find all the roots of $f(x)$ in $\mathbb{Z}_5$. How many are there?
   (b) Find all the roots of $f(x)$ in $\mathbb{Z}_{10}$. How many are there?
   (c) Explain why you can get so many roots for a quadratic polynomial in $\mathbb{Z}_{10}$.