Math 2111 Exam 2 Solutions

Nov 18, 2005

1. (5 pts each) Decide if the following statements are true or false. Prove those that are true and give a counterexample to those that are false.

   (a) Let F be a field and A an invertible $n \times n$ matrix over F. Then
   $$(A^T)^{-1} = (A^{-1})^T$$

   $$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
   $$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$
   Hence A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   (b) Let F be a field and $A, B \in M_{n\times n}(F)$. Then
   $$(A - B)(A + B) = A^2 - B^2.$$
   (Where A^2 and B^2 mean AA and BB.)

   $$(A - B)(A + B) = A^2 + AB - BA - B^2$$
   which is only equal to $A^2 - B^2$ if $AB - BA = 0$, that is if $AB = BA$. Since matrix multiplication is not commutative, it is easy to find two matrices A and B for which $(A - B)(A + B) \neq A^2 - B^2$. For example, let
   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
   Then
   $$(A - B)(A + B) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
   $$A^2 - B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

2. (10 pts each) Prove the following statements.

   (a) Let V be a (not necessarily finite dimensional) vector space and S, T subsets of V. If $S \subseteq T$ then $\text{span}(S) \subseteq \text{span}(T)$.

   Let $v \in \text{span}(S)$. Then v is a linear combination of the vectors in S. But any element of S is also in T, so this is a linear combination of vectors in T as well. Hence $v \in \text{span}(T)$. This is true for all $v \in \text{span}(S)$, so $\text{span}(S) \subseteq \text{span}(T)$.

   (b) Let V be a finite dimensional vector space, U a subspace of V and S a basis of U. There exists a basis T of V which contains S. (Hint: build T by extending S.)

   Since S is a basis of U, it is linearly independent. If S spans V, then S itself is a basis of V and we are done. If not, there exists $v \in V$ which is not in $\text{span}(S)$. Add v to the end of S. The set you get is still linearly independent because none of the vectors in it is a linear combination of the preceding ones. Continue this way until you get a set that spans V. Since $\dim(V)$ is finite, no linearly independent set in V can have more elements than $\dim(V)$, which is why this process must end eventually. The resulting set T spans V and is linearly independent, hence it is a basis of V. Obviously, $S \subseteq T$.

3. (10 pts) Let V be a (not necessarily finite dimensional) vector space and S a (not necessarily finite) subset of V. Prove that if every $v \in V$ can be expressed as a linear combination of vectors in S uniquely, then S is a basis of V.

   Every vector in V is a linear combination of vectors in S, therefore S spans V. Now, express $\vec{0}$ as a linear combination of vectors in S. This can be done in only one way, namely, the trivial way with all coefficients 0, hence S is linearly independent. So S is a basis of V.

4. (10 pts) Solve the following system of linear equations using row reduction. (You may use Gaussian elimination or Gauss-Jordan reduction, your choice.) Write your solution in vector form and verify that it is correct.

$$\begin{aligned} 2x_1 + x_2 + 5x_3 &= 13 \\ -x_1 + x_2 + 2x_3 &= -5 \\ 3x_1 - 2x_2 - 3x_3 &= 16 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 5 & 13 \\ -1 & 1 & 2 & -5 \\ 3 & -2 & -3 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -2 & 5 \\ 0 & 3 & 9 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -2 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence $x_3 = s \in \mathbb{R}$, $x_2 = 1 - 3s$, $x_1 = 6 - s$, or

$$\vec{x} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$\begin{aligned} 2(6-s) + (1-3s) + 5s &= 13 + 0s = 13 \quad \checkmark \\ -(6-s) + (1-3s) + 2s &= -5 + 0s = -5 \quad \checkmark \\ 3(6-s) - 2(1-3s) - 3s &= 16 + 0s = 16 \quad \checkmark \end{aligned}$$

5. (5 pts each) **Extra credit problem.**
   (a) Let $A, B \in M_{n\times n}(F)$ be symmetric matrices. Show that AB is a symmetric matrix if and only if $AB = BA$.

   We know $A^T = A$ and $B^T = B$. Using the usual properties of matrix multiplication and the transpose

$$(AB)^T = B^T A^T = BA$$

   So $(AB)^T = AB$ if and only if $AB = BA$, which is exactly what we wanted to prove.

   (b) Let $A, B, C \in M_{n\times n}(F)$. Is it true that if $AB = AC$, then $B = C$? If so, prove it, if not, give a counterexample.

   This is clearly false. For example, A could be the 0 matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   But A doesn't even have to be the 0 matrix for this. E.g.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$