Math 2111 Quiz Problems 1
Prepare by 9/19/05

1. Decide if the following are operations on the given sets. For those that are, say if they are commutative and/or associative. Is there an identity and if there is, what is it? If there is an identity, does every element have an inverse? Be sure to justify your answers.
   (a) $x \circ y = xy - 1$ on $\mathbb{Z}$,
   (b) $x \circ y = \sqrt{xy}$ on $\mathbb{Q}^+$ (the positive rational numbers),
   (c) $x \circ y = x\overline{y}$ on $\mathbb{C}$,
   (d) $x \circ y = x^2 y$ on $\mathbb{Z}_5$.
   (e) $x \circ y = (x + y)/2$ on $\mathbb{Q}$.
   (f) $x \circ y = 2^{xy}$ on $\mathbb{N}$.
   (g) $x \circ y = x^y$ on $\mathbb{N}$.
   (h) $f \circ g = fg$ on the set S of all functions $\mathbb{C} \to \mathbb{C}$. (Recall fg is defined by $fg(x) = f(x)g(x)$.)
2. Let $\circ$ be an operation on S. Prove that if $\circ$ has an identity, then this identity is unique (i.e. there cannot be two different identities).
3. Let $\circ$ be an associative operation on S with identity e. Prove that if an element x has an inverse, then this inverse is unique (i.e. an element cannot have two different inverses).
4. Let $\circ$ be an operation on the set S and let $T \subseteq S$. Prove or give a counterexample:
   (a) $\circ$ is also an operation on T.
   (b) If $\circ$ is associative on S and is an operation on T, then it is associative on T.
   (c) If $\circ$ is commutative on S and is an operation on T, then it is commutative on T.
   (d) If $\circ$ has an identity on S and is an operation on T, then it has an identity on T.
   (e) If $\circ$ is an operation on T and has an identity e on T, then e is also an identity on S.
   (f) If $\circ$ is an operation on T, has an identity e on S and $e \in T$, then e is an identity on T.
   (g) Suppose $\circ$ is an operation on T, e is an identity on S and $e \in T$. If every element in S has an inverse, then every element in T also has an inverse.
5. Let $\circ$ be an operation on S with identity e. Show that if
$$x \circ (y \circ z) = (x \circ z) \circ y \qquad \forall x, y, z \in S$$
   then $\circ$ is commutative and associative. (Hint: Prove commutativity first.)
6. Which of the following are fields and why or why not?
   (a) The set of all functions $\mathbb{R} \to \mathbb{R}$ with addition and multiplication defined by
$$(f + g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x).$$
   (b) The set of all functions $\mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$ with addition and multiplication as above.
   (c) $\mathbb{Z}[\sqrt{2}] = \{x + y\sqrt{2} \mid x, y, \in \mathbb{Z}\} \subseteq \mathbb{R}$ with addition and multiplication as real numbers.
   (d) $\mathbb{Q}[i] = \{x + yi \mid x, y, \in \mathbb{Q}\}$, where $i^2 = -1$ and addition and multiplication work just like over the complex numbers.
   (e) $\mathbb{R}[x]$, the set of all polynomials with real coefficients with the usual addition and multiplication.