Math 21D Solution Set 3

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4.5.6: The auxiliary equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$. General solution: $y(x) = C_1\, e^{2x} + C_2\, e^{3x}$.

4.5.14: The auxiliary equation is $r^2 + r = r(r+1) = 0$. General solution: $y(x) = C_1 + C_2\, e^{-x}$. From the initial conditions:

$$1 = y(0) = C_1 + C_2$$

$$1 = y'(0) = -C_2$$

Hence $C_1 = 2, C_2 = -1$, and $y(x) = 2 - e^{-x}$.

4.5.20: The auxiliary equation is $r^2 - 4r + 4 = (r-2)^2 = 0$. General solution: $y(x) = C_1\, e^{2x} + C_2\, xe^{2x}$. From the initial conditions:

$$1 = y(1) = C_1 e^2 + C_2 e^2$$

$$1 = y'(1) = 2C_1 e^2 + 3C_2 e^2$$

Hence $C_1 = 2e^{-2}, C_2 = -e^{-2}$, and $y(x) = 2\, e^{2x-2} - xe^{2x-2}$.

4.5.29: The auxiliary equation is $r^3 - 7r^2 + 7r + 15 = 0$. Notice that $r = -1$ is a root and the polynomial factors as $r^3 - 7r^2 + 7r + 15 = (r+1)(r^2 - 8r + 15)$. So the other two roots are 3 and 5. General solution: $y(x) = C_1\, e^{-x} + C_2\, e^{3x} + C_3\, e^{5x}$.

4.5.40: Let's assume $x > 0$ for now because the initial conditions both satisfy this. Then we can substitute $x = e^t$. Then $y(x) = y(e^t)$ and

$$\frac{d}{dt} y(x) = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx} e^t$$

and

$$\frac{d^2}{dt^2} y(x) = \frac{d^2y}{dx^2}\frac{dx}{dt} e^t + \frac{dy}{dx} e^t = \frac{d^2y}{dx^2} e^{2t} + \frac{dy}{dx} e^t.$$

Hence

$$\begin{aligned} x\frac{dy}{dx} &= \frac{d}{dt} y(e^t) \\ x^2\frac{d^2y}{dx^2} &= \frac{d^2}{dt^2} y(e^t) - \frac{d}{dt} y(e^t) \end{aligned}$$

So the equation turns into:

$$\frac{d^2}{dt^2} y(e^t) + 6\frac{d}{dt} y(e^t) + 5y(e^t) = 0$$

The auxiliary equation is $r^2 + 6r + 5 = (r+1)(r+5) = 0$. The general solution is $y(e^t) = C_1 e^{-t} + C_2 e^{-5t}$. Hence $y(x) = C_1 x^{-1} + C_2 x^{-5}$, which is a complete solution, because it never crosses the y-axis, and we don't have to worry about the case $x < 0$.

From the initial conditions:

$$-1 = y(1) = C_1 + C_2$$

$$13 = y'(1) = -C_1 - C_5$$

So $C_1 = 2, C_2 = -3$, and the particular solution is $y(x) = 2x^{-1} - 3x^{-5}$.

4.5.41: Since $x > 2$, $x - 2 > 0$ and we can let $x - 2 = e^t$. Then $y(x) = y(e^t + 2)$ and

$$\frac{d}{dt}y(x) = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t$$

and

$$\frac{d^2}{dt^2}y(x) = \frac{d^2y}{dx^2}\frac{dx}{dt}e^t + \frac{dy}{dx}e^t = \frac{d^2y}{dx^2}e^{2t} + \frac{dy}{dx}e^t.$$

Hence $x\frac{dy}{dx} = \frac{d}{dt}y(e^t + 2)$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2}{dt^2}y(e^t + 2) - \frac{d}{dt}y(e^t + 2)$. So the equation turns into:

$$\frac{d^2}{dt^2}y(e^t + 2) - 8\frac{d}{dt}y(e^t + 2) + 7y(e^t + 2) = 0$$

The auxiliary equation is $r^2 - 8r + 7 = (r - 1)(r - 7) = 0$. So the general solution is $y(e^t + 2) = C_1e^t + C_2e^{7t}$. Hence $y(x) = C_1(x - 2) + C_2(x - 2)^7$.

4.6.22: Auxiliary equation: $r^2 + 2r + 17 = 0$, and the roots are $-1 \pm 4i$. General solution: $y(x) = y(x) = e^{-x}\,(C_1\ \cos(4\,x) + C_2\ \sin(4\,x))$. From the initial conditions:

$$1 = y(0) = C_1$$

$$-1 = y'(0) = -C_1 + 4C_2$$

Hence $C_1 = 1, C_2 = 0$, and $y(x) = e^{-x}\cos(4x)$.

4.6.35: Let's first assume $x > 0$, and substitute $x = e^t$. Then $y(x) = y(e^t)$ and

$$\frac{d}{dt}y(x) = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t$$

and

$$\frac{d^2}{dt^2}y(x) = \frac{d^2y}{dx^2}\frac{dx}{dt}e^t + \frac{dy}{dx}e^t = \frac{d^2y}{dx^2}e^{2t} + \frac{dy}{dx}e^t.$$

$$\begin{aligned} x\frac{dy}{dx} &= \frac{d}{dt}y(e^t) \\ x^2\frac{d^2y}{dx^2} &= \frac{d^2}{dt^2}y(e^t) - \frac{d}{dt}y(e^t) \end{aligned}$$

So the equation turns into:

$$\frac{d^2}{dt^2}y(e^t) + 2\frac{d}{dt}y(e^t) + 5y(e^t) = 0$$

The auxiliary equation is $r^2 + 2r + 5 = 0$. The roots are $-1 \pm 2i$. So the general solution is $y(e^t) = e^{-t}(C_1\cos(2t) + C_2\sin(2t))$. Hence $y(x) = x^{-1}(C_1\cos(2\ln x) + C_2\sin(2\ln x))$ for $x > 0$.

If $x < 0$, substitute $x = -e^t$. Then $y(x) = y(-e^t)$ and

$$\frac{d}{dt}y(x) = \frac{dy}{dx}\frac{dx}{dt} = -\frac{dy}{dx}e^t$$

and

$$\frac{d^2}{dt^2}y(x) = -\frac{d^2y}{dx^2}\frac{dx}{dt}e^t - \frac{dy}{dx}e^t = \frac{d^2y}{dx^2}e^{2t} - \frac{dy}{dx}e^t.$$

Hence

$$\begin{aligned} x\frac{dy}{dx} &= \frac{d}{dt}y(e^t) \\ x^2\frac{d^2y}{dx^2} &= \frac{d^2}{dt^2}y(e^t) - \frac{d}{dt}y(e^t) \end{aligned}$$

So the equation turns into:

$$\frac{d^2}{dt^2}y(-e^t) + 2\frac{d}{dt}y(-e^t) + 5y(-e^t) = 0$$

The auxiliary equation is $r^2+2r+5=0$. The roots are $-1\pm 2i$. So the general solution is $y(-e^t) = e^{-t}(C_1\cos(2t) + C_2\sin(2t))$. Hence $y(x) = (-x)^{-1}(C_1\cos(2\ln(-x)) + C_2\sin(2\ln(-x)))$ for $x < 0$.

4.6.36: a. The equation is $10x''(t) + 250x(t) = 0$, or after dividing by 10, it is $x''(t) + 25x(t) = 0$. The auxiliary equation is $r^2 + 25 =$, and its roots are $\pm 5i$. So the general solution is $x(t) = C_1\cos(5t) + C_2\sin(5t)$. From the initial conditions

$$30 = x(0) = C_1$$

$$-10 = x'(0) = 5C_2$$

Hence $C_1 = 30$ cm, $C_2 = -2$ cm, and the particular solution is $x(t) = 30\cos(5t) - 2\sin(5t)$.

b. $\nu = \frac{5}{2\pi}$.

4.6.37: a. The equation is $10x''(t) + 60x'(t) + 250x(t) = 0$, or after dividing by 10, it is $x''(t) + +6x'(t) + 25x(t) = 0$. The auxiliary equation is $r^2 + 6r^2 + 25 =$, and its roots are $-3 \pm 4i$. So the general solution is $x(t) = e^{-3t}(C_1\cos(4t) + C_2\sin(4t))$. From the initial conditions

$$30 = x(0) = C_1$$

$$-10 = x'(0) = -3C_1 + 4C_2$$

Hence $C_1 = 30$ cm, $C_2 = 20$ cm, and the particular solution is $x(t) = e^{-3t}(30\cos(5t) + 20\sin(5t))$.

b. $\nu = \frac{4}{2\pi} = \frac{2}{\pi}$.

c. It reduces the frequency. It also introduces the decay factor of e^{-3t}, which makes the amplitude go to 0 exponentially with time.