Math 21D Solution Set 4

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4.7.6: The general solutions is the superposition of the particular solution and the general solution of the homogeneous equation $y'' + y = 0$. The auxiliary equation is $r^2 + 1 = (r + i)(r - i) = 0$. Hence the general solution is $y(x) = e^{2x} + C_1 \cos(x) + C_2 \sin(x)$.

4.7.9: Again, find the general solution of $y'' - 2y' + y = 0$ first. The auxiliary equation is $r^2 - 2r + 1 = (r - 1)^2 = 0$. Hence the general solution is $y(x) = x^2 e^x + C_1 x e^x + C_2 e^x$.

4.7.13.a):

$$\begin{aligned}
L[y_1](x) &= \frac{d^2}{dx^2}(1/2 \tan x) - \tan x = \tan x \left(1 + \tan^2 x\right) - \tan x = \tan^3 x \\
L[y_2](x) &= \frac{d^2}{dx^2}x - 2\,x = -2x \\
L[y_3](x) &= \frac{d^2}{dx^2}\left(-\frac{1}{2}\right) + 1 = 1
\end{aligned}$$

Notice that $\tan^3 x - 1 = L[y_1](x) - L[y_3](x) = L[y_1 - y_3](x)$. Hence the particular solution is $y_1(x) - y_3(x) = (\tan x + 1)/2$. The general solution of the homogeneous equation $L[y](x) = 0$ is $C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$ (the auxiliary equation is $r^2 - 2 = 0$). Hence the general solution of $L[y](x) = \tan^3 x + 1$ is $y(x) = (\tan x + 1)/2 + C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$.

4.7.14.a):

$$\begin{aligned}
L[y_1](x) &= \frac{d^2}{dx^2}x - 4\,\frac{d}{dx}x + 3\,x = 3x - 4 \\
L[y_2](x) &= \frac{d^2}{dx^2}e^{-x} - 4\,\frac{d}{dx}e^{-x} + 3\,e^{-x} = 8\,e^{-x}
\end{aligned}$$

Notice that $8e^{-x} + 8 - 6x = -2L[y_1](x) - L[y_2](x) = L[-2y_1 - y_3](x)$. Hence the particular solution is $-2y_1(x) + y_3(x) = e^{-x} - 2x$. The general solution of the homogeneous equation $L[y](x) = 0$ is $C_1 e^x + C_2 e^{3x}$ (the auxiliary equation is $r^2 - 4r + 3 = (r - 1)(r - 3) = 0$). Hence the general solution of $L[y](x) = 8e^{-x} + 8 - 6x$ is $y(x) = e^{-x} - 2x + C_1 e^x + C_2 e^{3x}$.

The initial conditions force $2 = y(0) = 1 + C_1 + C_2$, and $-2 = y'(0) = -3 + C_1 + 3C_2$. Hence $C_1 = 1$, $C_2 = 0$, and $y(x) = e^x + e^{-x} - 2x$.

4.7.19: a. To solve $v' + v = e^x$, multiply by $\mu(x) = e^{\int 1 dx} = e^x$. Then

$$\begin{aligned}
e^x v' + e^x v &= e^{2x} \\
\frac{d}{dx}(e^x v) &= e^{2x} \\
e^x v &= \frac{e^{2x}}{2} + C_1 \\
v(x) &= \frac{e^x}{2} + C_1 e^{-x}
\end{aligned}$$

b. To solve $y' - 2y = \frac{e^x}{2} + C_1 e^{-x}$, multiply by $\mu(x) = e^{\int -2dx} = e^{-2x}$:

$$\begin{aligned}
e^{-2x}y' + e^{-2x}y &= \frac{e^{-x}}{2} + C_1 e^{-3x} \\
\frac{d}{dx}(e^{-2x}y) &= \frac{e^{-x}}{2} + C_1 e^{-3x} \\
e^{-2x}y &= -\frac{e^{-x}}{2} - C_1\frac{e^{-3x}}{3} + C_2 \\
y(x) &= -\frac{e^{x}}{2} - \frac{C_1}{3}e^{-x} + C_2 e^{2x}
\end{aligned}$$

Since $-C_1/3$ is just another constant, you can replace it with C_3, and the final answer is $y(x) = -e^x/2 + C_3 e^{-x} + C_2 e^{2x}$.

4.7.20: Notice that $D^2 + 6D + 5 = (D+1)(D+5)$. Hence we can can solve the two equations:

$$\begin{aligned}
(D+5)v &= 10x + 5 \\
(D+1)y &= v
\end{aligned}$$

To solve $v' + 5v = 10x + 5$, multiply by $\mu(x) = e^{\int 5dx} = e^{5x}$. Then

$$\begin{aligned}
e^{5x}v' + e^{5x}v &= e^{5x}(10x+5) \\
\frac{d}{dx}(e^{5x}v) &= e^{5x}(10x+5) \\
e^{5x}v &= 10\int xe^{5x}dx + 5\int e^{5x}dx = 2xe^{5x} + 3\int e^{5x}dx = 2xe^{5x} + \frac{3}{5}e^{5x} + C_1 \\
v(x) &= 2x + \frac{3}{5} + C_1 e^{-5x}
\end{aligned}$$

To solve $y' + y = 2x + \frac{3}{5} + C_1 e^{-5x}$, multiply by $\mu(x) = e^{\int 1dx} = e^x$:

$$\begin{aligned}
e^x y' + e^x y &= 2xe^x + \frac{3}{5}e^x + C_1 e^{-4x} \\
\frac{d}{dx}(e^x y) &= 2xe^x + \frac{3}{5}e^x + C_1 e^{-4x} \\
e^x y &= \int 2xe^x dx + \frac{3}{5}\int e^x dx + C_1\int e^{-4x}dx = 2xe^x - \int 2e^x dx + \frac{3}{5}e^x - C_1\frac{e^{-4x}}{4} \\
y(x) &= 2x - \frac{7}{5} - \frac{C_1}{4}e^{-5x} + C_2 e^{-x}
\end{aligned}$$

Since $-C_1/4$ is just a constant, you can replace it with C_3, and the final answer is $y(x) = 2x - 7/5 + C_3 e^{-5x} + C_2 e^{-x}$.