Math 21D solution Set 5
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4.8.11: Guess $y(x) = A2^x$. Then

$$\begin{aligned} A2^x (\ln 2)^2 + A2^x &= 2^x \\ A(\ln 2)^2 + A &= 1 \end{aligned}$$

$$A = \left((\ln 2)^2 + 1\right)^{-1}$$

So a particular solution is $y(x) = \left((\ln 2)^2 + 1\right)^{-1} 2^x$.

4.8.21: Let's first find a particular solution. Guess $y(t) = Ae^t \sin(t) + Be^t \cos(t)$. Then

$$L(y) = -Ae^t \cos t + Be^t \sin t - Ae^t \sin t - Be^t \cos t = e^t - \sin t$$

Hence

$$\begin{aligned} -A - B &= 0 \\ B - A &= 1 \end{aligned}$$

which is satisfied by $A = -1/2$ and $B = 1/2$. Since $D^2 - 3D + 2 = (D-1)(D-2)$, the general solution is

$$y(t) = 1/2\,\cos t e^t - 1/2 e^t \sin t + C_1 e^t + x C_2 e^{2t}$$

4.7.36: Note that $L(e^\theta) = 0$ and $L(e^{-\theta}) = 0$. Hence a particular solution is of the form $y(\theta) = \theta\left(Ae^\theta + Be^{-\theta}\right) + C$. Then

$$L(y) = 2\,Ae^\theta - 2\,Be^{-\theta} - C = e^\theta - e^{-\theta} + 2$$

which gives $A = 1/2$, $B = 1/2$, and $C = -2$. Notice that $D^2 - 1 = (D-1)(D+1)$, so the general solution is $y(\theta) = \theta\left(1/2e^\theta + 1/2e^{-\theta}\right) - 2 + C_1 e^\theta + C_2 e^{-\theta}$. Use the initial conditions to obtain

$$\begin{aligned} 0 = y(0) &= C_1 + C_2 - 2 \\ 0 = y'(0) &= 1 + C_1 - C_2 \end{aligned}$$

which yields $C_1 = 1/2$ and $C_2 = 3/2$. So the solution is

$$y(\theta) = 1/2\theta\, e^\theta + 1/2\theta\, e^{-\theta} - 2 + 1/2e^\theta + 3/2e^{-\theta}$$

4.8.48: Note that $L(e^{-x}\cos(x)) = 0$ and $L(e^{-x}\sin(x)) = 0$. Use the rules on p. 197 to conclude that there is a particular solution of the form

$$y(x) = C_1 x^2 + C_2 x + C_3 + x((C_4 x + C_5)e^{-x}\cos(x) + (C_6 x + C_7)e^{-x}\sin(x))$$

If you actually computed the general solution, you'd find

$$y(x) = 1/2\,x^2 - x + 1/2 + 3/4\,xe^{-x}\sin x - 1/4\,x^2 e^{-x}\cos x + C_1\,e^{-x}\sin x + C_2\,e^{-x}\cos x$$

4.8.57: Note that $L(e^x) = 0$. So we need to raise the power of x in the e^x term of the particular solution. Guess $y(x) = x\,(Axe^x + Be^x) + C$. Then

$$L(y) = 8\,Ae^x + 10\,Axe^x + 5\,Be^x - 2\,C = xe^x + 1$$

gives $A = 1/10$, $B = -4/25$, and $C = -1/2$. So a particular solution is

$$y(x) = 1/10x^2e^x - 4/25xe^x - 1/2$$

4.8.61: First, divide the equation by x^2 (you know $x^2 \neq 0$) to obtain $y''(x) - 6y(x) = x^{-2} - 6$, which is now a Cauchy-Euler equation. After substituting $x = e^t$, we have

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = e^{-2t} - 6$$

Note that $D^2 - D - 6 = (D-3)(D+2)$, so the solutions of the homogeneous equation are $y(t) = C_1e^{3t} + C_2e^{-2t}$. Hence $L(e^{-2t}) = 0$, and you should guess $y(t) = Ate^{-2t} + B$ for the particular solution. Then

$$L(y) = -5\,Ae^{-2\,t} - 6\,B = e^{-2t} - 6$$

gives $A = -1/5$ and $B = 1$. So the general solution is $y(t) = -1/5te^{-2t} + 1 + C_1e^{3t} + C_2e^{-2t}$. But we want it all in terms of x. That is

$$y(x) = -1/5x^{-2}\ln x + 1 + C_1x^3 + C_2x^{-2}$$

4.9.1: $D^2 + 4 = (D+2i)(D-2i)$, so the solutions of the homegeneous equation are linear combinations of $y_1 = \cos 2x$ and $y_2 = \sin 2x$.

Let $y = v_1y_1 + v_2y_2$, and use variation of parameters as in 4.9:

$$\begin{aligned} v_1'y_1 + v_2'y_2 = v_1'\cos 2x + v_2'\sin 2x &= 0 \\ v_1'y_1' + v_2'y_2' = -v_1'2\sin 2x + v_2'2\cos 2x &= \tan 2x \end{aligned}$$

Add $(2\sin 2x)(1) + (\cos 2x)(2)$:

$$\begin{aligned} v_2'(2\sin^2 2x + 2\cos^2 2x) &= \sin 2x \\ v_2' &= \frac{\sin 2x}{2} \\ v_1' &= -\frac{\sin^2 2x}{2\cos 2x} \end{aligned}$$

Now integrate to find v_1 and v_2:

$$\begin{aligned} v_2 &= \int \frac{\sin 2x}{2}dx = -\frac{\cos 2x}{4} \\ v_1 &= -\int \frac{\sin^2 2x}{2\cos 2x}dx = \int \frac{\cos^2 2x - 1}{2\cos 2x}dx = \frac{1}{2}\int \cos 2x - \sec 2x dx \\ &= \frac{1}{4}(\sin 2x - \ln|\sec 2x + \tan 2x|) \end{aligned}$$

Hence a particular solution is $y(x) = \frac{1}{4}((\sin 2x - \ln|\sec 2x + \tan 2x|)\cos 2x - \cos 2x \sin 2x) = -\frac{1}{4}\ln|\sec 2x + \tan 2x|\cos 2x$, and the general solution is

$$y(x) = \frac{1}{4}\ln|\sec 2x + \tan 2x|\cos 2x + C_1\cos 2x + C_2\sin 2x$$

4.9.22: First divide by x^2 (you know it's not 0):

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = x^{1/2}$$

Now let $y = v_1 y_1 + v_2 y_2$ and use variation of parameters:

$$\begin{aligned}
v_1' y_1 + v_2' y_2 = v_1' x^{-1/2} \cos x + v_2' x^{-1/2} \sin x &= 0 \\
v_1' y_1' + v_2' y_2' = v_1'(-1/2 x^{-3/2} \cos x - x^{-1/2} \sin x) + v_2'(-1/2 x^{-3/2} \sin x + x^{-1/2} \cos x) &= x^{1/2}
\end{aligned}$$

Multiply the first equation by $x^{1/2}$ and the second by $x^{3/2}$ to simplify them.

$$\begin{aligned}
v_1' y_1 + v_2' y_2 = v_1' \cos x + v_2' \sin x &= 0 \\
v_1' y_1' + v_2' y_2' = -1/2 v_1' \cos x - v_1' x \sin x - 1/2 v_2' \sin x + v_2' x \cos x &= x^2
\end{aligned}$$

Notice that $-1/2 v_1' \cos x - 1/2 v_2' \sin x = 0$, so the second equation turns into $-v_1' x \sin x + v_2' x \cos x = x^2$. After dividing by x, we have $-v_1' \sin x + v_2' \cos x = x$. It is now easy to compute

$$\begin{aligned}
v_2' &= x \cos x \\
v_1' &= -x \sin x
\end{aligned}$$

After integration by parts

$$\begin{aligned}
v_2 &= x \sin x + \cos x \\
v_1 &= x \cos x - \sin x
\end{aligned}$$

A particular solution is $y(x) = (x \cos x - \sin x) x^{-1/2} \cos x + (x \sin x + \cos x) x^{-1/2} \sin x = x^{1/2}$. And the general solution is $y(x) = x^{1/2} + C_1 x^{-1/2} \cos x + C_2 x^{-1/2} \sin x$.

4.9.5: Notice that $D^2 - 2D + 1 = (D-1)^2$. Hence we can solve the two equations:

$$\begin{aligned}
(D-1)v &= x^{-1} e^x \\
(D-1)y &= v
\end{aligned}$$

To solve $v' - v = x^{-1} e^x$, multiply by $\mu(x) = e^{\int -1 dx} = e^{-x}$. Then

$$\begin{aligned}
e^{-x} v' - e^{-x} v &= 1/x \\
\frac{d}{dx}(e^x v) &= 1/x \\
e^x v &= \int 1/x dx = \ln|x| + C_1 \\
v(x) &= e^x \ln|x| + C_1 e^x
\end{aligned}$$

To solve $y' - y = e^x \ln|x| + C_1 e^x$, multiply by $\mu(x) = e^{-x}$:

$$\begin{aligned}
e^{-x} y' - e^{-x} y &= \ln|x| + C_1 \\
\frac{d}{dx}(e^{-x} y) &= \ln|x| + C_1 \\
e^{-x} y &= \int \ln|x| + C_1 dx = x \ln|x| - x + C_1 x + C_2 \\
y(x) &= x e^x \ln|x| - x e^x + C_1 x e^x + C_2 e^x
\end{aligned}$$

You can integrate $\ln|x|$ by noting that $\ln|x| = \ln x$ for $x > 0$ and $\ln|x| = \ln(-x)$ for $x < 0$. Hence $\int \ln|x| dx = \int \ln x dx = x \ln x - x + c$ for $x > 0$ and $\int \ln|x| dx = \int \ln(-x) dx = -((-x)\ln(-x) - (-x)) + c = x \ln(-x) - x + c$ for $x < 0$. Hence $\int \ln|x| dx = x \ln|x| - x + c$.