Math 21D solution Set 6

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5.1.8: The two equations to solve are

$$\begin{aligned} \frac{2\pi}{3} &= \sqrt{\frac{k}{m}} \\ \frac{2\pi}{4} &= \sqrt{\frac{k}{m+2}} \end{aligned}$$

Divide the first by the second:

$$\begin{aligned} \frac{4}{3} &= \sqrt{\frac{m+2}{m}} \\ \frac{16}{9} &= 1+\frac{2}{m} \\ m &= \frac{18}{7}\text{ lb} \end{aligned}$$

5.3.9: a. We have to solve

$$mx'' + kx = F_0\cos(\gamma t)$$

The auxiliary equation is $mr^2 + kr = 0$, so the homogeneous solutions are $x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$ where $\omega = \sqrt{k/m}$. Since $\gamma \neq \omega$, we can look for a particular solution of the form $x_p = A\cos(\gamma t) + B\sin(\gamma t)$. Then

$$\begin{aligned} F_0\cos(\gamma t) &= mx_p'' + kx_p \\ &= -mA\gamma^2\cos(\gamma t) - mB\gamma^2\sin(\gamma t) + kA\cos(\gamma t) + B\sin(\gamma t) \end{aligned}$$

Hence

$$\begin{aligned} A(-m\gamma^2 + k) &= F_0 \\ B(-m\gamma^2 + k) &= 0 \end{aligned}$$

So $A = \frac{F_0}{k-m\gamma^2}$, and $B = 0$. So $x(t) = \frac{F_0}{k-m\gamma^2}\cos(\gamma t) + C_1\cos(\omega t) + C_2\sin(\omega t)$. Use the initial conditions to find

$$\begin{aligned} 0 = x(0) &= \frac{F_0}{k-m\gamma^2} + C_1 \\ 0 = x'(0) &= C_2\omega \end{aligned}$$

hence $C_1 = -\frac{F_0}{k-m\gamma^2}$ and $C_2 = 0$.

$$x(t) = \frac{F_0}{k-m\gamma^2}(\cos(\gamma t) - \cos(\omega t)).$$

b. Notice that $k = \omega^2 m$, so

$$\frac{F_0}{k-m\gamma^2} = \frac{F_0}{m(\omega^2-\gamma^2)}$$

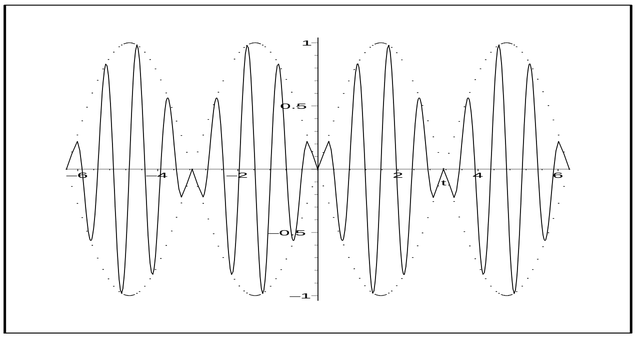
Now use

$$\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$$

to derive

$$x(t) = \frac{2F_0}{m(\omega^2-\gamma^2)}\sin\left(\frac{\gamma+\omega}{2}t\right)\sin\left(\frac{\omega-\gamma}{2}t\right).$$

c.



8.2.2: Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}n!}{(n+1)!3^n} = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

So $\rho = \infty$, and the covergence set is all of $\mathbb{R}$.

8.2.5: Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3n^3}{(n+1)^3 3} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = 1$$

So $\rho = 1$. So the series certainly converges for $1 < x < 3$.

Let's look at the endpoints. If $x = 3$, then

$$\sum_{n=1}^{\infty}\frac{3}{n^3} = 3\sum_{n=1}^{\infty}\frac{1}{n^3}$$

which is well-known to be convergent. Note that $\sum_{n=2}^{\infty} \leq \int_1^{\infty} x^{-3}dx = 1/2$ because it's a right-hand sum. Hence $\sum_{n=1}^{\infty}\frac{1}{n^3} \leq 3/2$.

If $x = 1$, then

$$\sum_{n=1}^{\infty}\frac{3}{n^3}(-1)^n = 3\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

But

$$-3/2 \leq \sum_{n=1}^{\infty} -\frac{1}{n^3} < \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} < \sum_{n=1}^{\infty}\frac{1}{n^3} \leq 3/2$$

Hence the convergence set includes the endpoints. It is $[1, 3]$.

8.2.7: Substitute $z = x^2$ and $b_k = a_{2k}$:

$$\sum_{k=0}^{\infty} a_{2k}x^{2k} = \sum_{k=0}^{\infty} b_k z^k$$

Use the ratio test on the latter:

$$L = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_{2n+2}}{a_{2n}}\right|$$

Then $\rho = 1/L$, and the series is convergent if $|z| < \rho$. In terms of x, the series converges if $|x| < \sqrt{\rho} = 1/\sqrt{L}$.

Notice that $\sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} = x\sum_{k=0}^{\infty} a_{2k+1}x^{2k}$. Hence $\sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}$ is convergent if and only if $\sum_{k=0}^{\infty} a_{2k+1}x^{2k}$ is. Now substitute $z = x^2$ and $b_k = a_{2k+1}$:

$$M = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_{2n+3}}{a_{2n+1}}\right|$$

Then $\rho = 1/M$, and the series is convergent if $|z| < \rho$. In terms of x, the series converges if $|x| < \sqrt{\rho} = 1/\sqrt{M}$.

8.2.8.d: Use the result of 8.2.7:

$$L = \lim_{n\to\infty}\left|\frac{a_{2n+2}}{a_{2n}}\right| = \lim_{n\to\infty}\frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0$$

Hence $\rho = \infty$, and the convergence set is $\mathbb{R}$.

8.2.8.f: Similarly to 8.2.7, substitute $z = x^4$, and $b_k = 4^k$. Then

$$L = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{4^{k+1}}{4^k} = 4$$

Then $\rho = 1/4$, and the series is convergent if $|z| < 1/4$. In terms of x, the series converges if $|x^4| < 1/4$, or $|x| < 1/\sqrt{2}$.

Let's check the endpoints of this interval. If $x = 1/sqrt2$, then the series is

$$\sum_{k=0}^{\infty} 2^{2k}(1/\sqrt{2})^{4k} = \sum_{k=0}^{\infty} 1$$

which is divergent. Since $(-1/\sqrt{2})^{4k} = (1/\sqrt{2})^{4k}$, it's the same when $x = -1/sqrt2$. So the convergence set is $(-1/\sqrt{2}, 1/\sqrt{2})$.

8.2.12:

$$\sin x\cos x = x - \left(\frac{1}{3!}+\frac{1}{2!}\right)x^3 + \left(\frac{1}{5!}+\frac{1}{2!3!}+\frac{1}{4!}\right)x^5 - \ldots = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \ldots$$

Using $\sin x\cos x = 1/2\sin(2x)$:

$$\begin{aligned}\sin x\cos x &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(2x)^{2k+1}}{(2k+1)!}\\ &= \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right) = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \ldots\end{aligned}$$

8.2.22:

$$g(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{x^{2k+1}}{2k+1}$$

8.2.28:

$$\begin{aligned} 2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n b_n x^{n-1} &= 2\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} (n+1) b_{n+1} x^n \\ &= b_1 + \sum_{n=1}^{\infty} \left(2a_{n-1} + (n+1)b_{n+1}\right) x^n \end{aligned}$$

8.2.31: Remember that the Taylor expansion of $(1-x)^{-1}$ about $x_0 = 0$ is $\sum_{n=0}^{\infty} x^n$. Hence

$$\frac{1+x}{1-x} = (1+x)\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} x^n = 1 + 2\sum_{n=1}^{\infty} x^n$$

8.2.38: Let $y = \sum_{n=0}^{\infty} x^n$. Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} a_n n x^{n-1} \\ y^2 &= \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 \end{aligned}$$

The coefficients of x^i in these two sums must be equal for $y' = y^2$. The first three nonzero terms of these series are

$$\begin{aligned} y' &= a_1 + 2a_2 x + 3a_3 x^2 + \ldots \\ y^2 &= a_0^2 + 2a_0 a_1 x + (a_1^2 + 2a_0 a_2)x^2 + \ldots \end{aligned}$$

This gives the system of equations

$$\begin{aligned} a_1 &= a_0^2 \\ 2a_2 &= 2a_0 a_1 \\ 3a_3 &= a_1^2 + 2a_0 a_2 \end{aligned}$$

From the initial condition $y(0) = 1$, we have $a_0 = 1$. Hence $a_1 = a_0^2 = 1$, $a_2 = a_0 a_1 = 1$, and $a_3 = (a_1^2 + 2a_0 a_2)/3 = 1$. So $y(x) = 1 + x + x^2 + x^3 + \ldots$.

Notice that this is a separable linear equation:

$$\begin{aligned} \frac{dy}{dx} &= y^2 \\ \int \frac{dy}{y^2} &= \int dx \qquad\qquad \text{We know } y(0) = 1 \Rightarrow y \neq 0 \\ -y^{-1} &= x + c \\ y &= \frac{-1}{x+c} \end{aligned}$$

From $1 = y(0) = -1/c$, $c = -1$. Hence $y(x) = 1/(1-x)$.

Note that $1/(1-x) = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$, so the first terms of the series above match this solution.