Math 21D solution Set 7

Imre Tuba

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8.3.4: The singular points are where

$$\frac{3}{x^2+x} \quad \text{or} \quad \frac{-6x}{x^2+x}$$

have a singularity. The first is singular whenever $x^2+x=0$, that is at 0 and -1. The second expression has no singularities in addition to these two, hence the set of singular points is $\{-1,0\}$.

8.3.15: Let $y=\sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Hence

$$\begin{aligned}
0 &= y'' + (x-1)y' + y \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + (x-1)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n \\
&= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n\right)x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} \\
&= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n\right)x^n + \sum_{n=1}^{\infty} n a_n x^n \\
&= (2a_2 - a_1 + a_0) + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + (n+1)a_n\right)x^n
\end{aligned}$$

Since there are no initial conditions, we expect that the general solution is the combination of two linearly independent solutions. Hence a_0 and a_1 remain free parameters in the solution. We get the following equations for a_2 and a_3:

$$\begin{aligned}
2a_2 - a_1 + a_0 &= 0 \\
6a_3 - 2a_2 + 2a_1 &= 0
\end{aligned}$$

from which $a_2 = 1/2(a_1 - a_0)$ and $a_3 = 1/3(a_2 - a_1) = -1/6(a_0 + a_1)$. So the first four terms of y are

$$y(x) = a_0 + a_1 x + \frac{a_1 - a_0}{2}x^2 - \frac{a_0 + a_1}{6}x^3 + \dots$$

8.3.24: Just like in the previous problem, $y = \sum_{n=0}^{\infty} a_n x^n$,

$$-xy' = -x\sum_{n=1}^{\infty} na_n x^{n-1} = -\sum_{n=1}^{\infty} na_n x^n$$

and

$$(x^2+1)y'' = (x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Hence

$$\begin{aligned}
0 &= (x^2+1)y'' - xy' + y \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= (2a_2 + a_0) + (6a_3 - a_1 + a_1)x \\
&\quad + \sum_{n=2}^{\infty}\left((n^2-n)a_n + (n+2)(n+1)a_{n+2} - na_n + a_n\right)x^n \\
&= (a_0 + 2a_2) + 6a_3 x + \sum_{n=2}^{\infty}\left((n-1)^2 a_n + (n+2)(n+1)a_{n+2}\right)x^n
\end{aligned}$$

Since there are no initial conditions, we expect that the general solution is the combination of two linearly independent solutions. Hence a_0 and a_1 remain free parameters in the solution. We get the following equations for a_2 and a_3 from the first two terms:

$$\begin{aligned}
a_0 + 2a_2 &= 0 \\
6a_3 &= 0
\end{aligned}$$

from which $a_2 = -1/2a_0$ and $a_3 = 0$. The recursive relation that determines the rest of the coefficients is

$$\begin{aligned}
(n-1)^2 a_n + (n+2)(n+1)a_{n+2} &= 0 \\
a_{n+2} &= -\frac{(n-1)^2}{(n+2)(n+1)}a_n
\end{aligned}$$

Since $a_3 = 0$, we immediately see that $a_n = 0$ whenever n is odd. When n is even, the closed form of the expression for a_n is easily seen to be

$$a_n = -(-1)^{n/2}\frac{(1\cdot 3\cdot 5\cdots(n-3))^2}{n(n-1)\cdots 3}a_2 = (-1)^{n/2}\frac{(1\cdot 3\cdot 5\cdots(n-3))^2}{n!}a_0$$

Hence the general solution of the differential equation is

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \sum_{k=2}^{\infty}(-1)^k\frac{(1\cdot 3\cdot 5\cdots(2k-3))^2}{(2k)!}x^{2k}\right) + a_1 x$$

8.3.34: Let $y = \sum_{n=0}^{\infty} c_n x^n$. We know from the initial conditions that $c_0 = 1$ and $c_1 = 0$. Hence we can write $y = 1 + \sum_{n=2}^{\infty} c_n x^n$. Then

$$\begin{aligned}
\frac{2}{x}y' &= \frac{2}{x}\sum_{n=2}^{\infty} nc_n x^{n-1} = 2\sum_{n=2}^{\infty} nc_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)c_{n+2}x^n \\
y'' &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n \\
y^n &= (1 + c_2x^2 + c_3x^3 + c_4x^4 + \cdots)^n = 1 + nc_2x^2 + nc_3x^3 + \binom{n}{2}c_2^2x^4 + \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= y'' + \frac{2}{x}y' + y^n \\
&= (2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots) + (4c_2 + 6c_3x + 8c_4x^2 + 10c_5x^3 + \cdots) \\
&\quad +(1 + nc_2x^2 + nc_3x^3 + \cdots) \\
&= 6\,c_2 + 1 + 12\,c_3x + (20\,c_4 + nc_2)\,x^2 + (30\,c_5 + nc_3)\,x^3 + \cdots
\end{aligned}$$

which gives us the following linear equations:

$$\begin{aligned}
6c_2 + 1 &= 1 \\
12c_3 &= 0 \\
20c_4 + nc_2 &= 0
\end{aligned}$$

The solution is $c_2 = -1/6$, $c_3 = 0$, and $c_4 = -nc_2/20 = n/120$. So

$$y(x) = 1 - \frac{x^2}{6} + \frac{nx^4}{120} + \cdots$$

which agrees with the solution in the book.

8.3.36: You know $x(t) = 3 + c_2t^2 + c_3t^3 + \cdots$ from the initial conditions. From here on, it's the usual game:

$$\begin{aligned}
0 &= 2x'' + x' + (6-t)x \\
&= 2(2c_2 + 6c_3t + 12c_4t^2 + \cdots) + (2c_2t + 3c_3t^2 + \cdots) \\
&\quad +6(3 + c_2t^2 + \cdots) - (t + c_2t^3 + \cdots) \\
&= 4\,c_2 + 18 + (12\,c_3 + 2\,c_2 - 3)\,t + (24\,c_4 + 3\,c_3 + 6\,c_2)\,t^2 + \cdots
\end{aligned}$$

which gives $c_2 = -9/2$, $c_3 = 1$, and $c_4 = 1$, and

$$x(t) = 3 - \frac{9}{2}t^2 + t^3 + t^4 + \cdots$$

8.4.9: Since we will have to deal with power series in $x - 1$, we might as well already substitute $t = x - 1$ to make life easy. Then $x^2 - 2x = x(x-2) = (t+1)(t-1) = t^2 - 1$, and the equation turns into $(t^2 - 1)y'' + 2y = 0$. Now do the usual $y = \sum_{n=0}^{\infty} a_n t^n$.

Then

$$\begin{aligned}
0=(t^2-1)y''+2y &= (t^2-1)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 2\sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n t^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty} 2a_n t^n \\
&= -2\,a_2 + 2\,a_0 + (-6\,a_3 + 2\,a_1)\,t + (4\,a_2 - 12\,a_4)\,t^2 + \cdots
\end{aligned}$$

which gives $a_2 = a_0$, $a_3 = a_1/3$. Hence the first four terms of the general solution, after substituting $t = x - 1$ back, are

$$y(t) = a_0 + a_1(x-1) + a_0(x-1)^2 + a_1/3(x-1)^3 + \cdots$$

8.4.14: Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned}
0 &= y' - e^x y \\
&= (a_1 + 2a_2x^2 + 3a_3x^2 + \cdots) \\
&\quad - \left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) \\
&= a_1 - a_0 + (2\,a_2 - a_0 - a_1)\,t + (3\,a_3 - 1/2\,a_0 - a_1 - a_2)\,t^2 + \cdots
\end{aligned}$$

from which

$$\begin{aligned}
a_1 - a_0 &= 0 \\
2\,a_2 - a_0 - a_1 &= 0 \\
3\,a_3 - 1/2\,a_0 - a_1 - a_2 &= 0
\end{aligned}$$

We are solving a first order linear differential equation, so we exect that the general soltuion will contain one free parameter. The solutions are $a_1 = a_0$, $a_2 = a_0$, and $a_3 = 5/6a_0$, and

$$y(x) = a_0 + a_0x + a_0x^2 + \frac{5a_0}{6}x^3 + \cdots$$

8.4.22: Let $w = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
a_1 + (2\,a_2 + a_0)\,x + (3\,a_3 + a_1)\,x^2 + (4\,a_4 + a_2)\,x^3 + \cdots &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots
\end{aligned}$$

gives

$$\begin{aligned}
a_1 &= 1 \\
2a_2 + a_0 &= 1 \\
3a_3 + a_1 &= \frac{1}{2}
\end{aligned}$$

The solution is $a_1 = 1$, $a_2 = (1 - a_0)/2$, and $a_3 = -1/6$. Hence

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{1}{6}x^3 + \cdots$$

8.4.29: a. Let $y = \sum_{k=0}^{\infty} a_k x^k$. Then

$$\begin{aligned}
0 &= (1-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)a_k x^k - 2\sum_{k=1}^{\infty} k a_k x^k + n(n+1)\sum_{k=0}^{\infty} a_k x^k \\
&= (2a_2 + n(n+1)a_0) + (6a_3 - 2a_1 + n(n+1)a_1)\,x \\
&\quad + \sum_{k=2}^{\infty}\left((k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + n(n+1)a_k\right)x^k \\
&= (2a_2 + n(n+1)a_0) + (6a_3 - 2a_1 + n(n+1)a_1)\,x \\
&\quad + \sum_{k=2}^{\infty}\left((k+2)(k+1)a_{k+2} + (n(n+1) - k(k+1))\,a_k\right)x^k
\end{aligned}$$

Let $N = n(n+1)$. The above equation gives

$$a_2 = -\frac{N}{2}a_0 \text{ and } a_3 = \frac{2-N}{6}a_1$$

and the recursion

$$a_{k+2} = \frac{k(k+1)-N}{(k+2)(k+1)}a_k$$

If k is even, let $k+2 = 2j$. Then the closed form of the solution is

$$\begin{aligned}
a_{2j} &= \frac{((2j-2)(2j-1)-N)((2j-4)(2j-3)-N)\cdots(2\cdot 3-N)a_2}{(2j)(2j-1)(2j-2)\cdots 3} \\
&= \frac{((2j-2)(2j-1)-N)((2j-4)(2j-3)-N)\cdots(2\cdot 3-N)(-N)a_0}{(2j)(2j-1)\cdots 2}
\end{aligned}$$

and for k odd let $k+2 = 2j+1$, that is $k = 2j-1$:

$$\begin{aligned}
a_{2j+1} &= \frac{((2j-1)(2j)-N)((2j-3)(2j-2)-N)\cdots(3\cdot 4-N)a_3}{(2j-1)(2j-2)(2j-3)\cdots 4} \\
&= \frac{((2j-1)(2j)-N)((2j-3)(2j-2)-N)\cdots(3\cdot 4-N)(1\cdot 2-N)a_1}{(2j-1)(2j-2)\cdots 3\cdot 2}
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y(x) &= a_0\left(1 + \sum_{j=1}^{\infty}\frac{((2j-2)(2j-1)-N)((2j-4)(2j-3)-N)\cdots(-N)}{(2j)!}x^{2j}\right) \\
&\quad + a_1\left(x + \sum_{j=1}^{\infty}\frac{((2j-1)(2j)-N)((2j-3)(2j-2)-N)\cdots(1\cdot 2-N)}{(2j-1)!}x^{2j+1}\right)
\end{aligned}$$

b. Notice that if $k = n$, then $k(k+1) - N = k(k+1) - n(n+1) = 0$. Hence $a_{n+2} = 0$, and consequently, $a_{n+2j} = 0$ for any $j > 0$. So one of the two recursions above always ends with zeroes. We can kill the other by choosing either $a_0 = 0$ (if n

is odd) or $a_1 = 0$ (if n is even). Now, choose $a_1 = 1$ if n is odd and $a_0 = 1$ if n is even. Then the expressions derived in part (a) yield a nonzero power series whose coefficients are all 0 starting with the x^{n+2} term. A finite power series is a polynomial. Since the degree of the last nonzero term is n, this polynomial is of degree n.

c. Use the formulas from part a, and the choices in part b:

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= x \\ L_2(x) &= 1 - 3x^2 \end{aligned}$$