Math 21D solution Set 8

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8.6.12: Since $p(x) = 4/x$ and $q(x) = 2/x^2$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 4 = 4$$

and

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} 2 = 2$$

So the auxiliary equation is $r(r-1)+4r+2 = r^2+3r+2 = (r+2)(r+1)$, and the exponents are $r_1 = -1$ and $r_2 = -2$.

8.6.27: So $p(x) = -1/x$ and $q(x) = -1$. Hence

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} -1 = -1$$

and

$$q_0 = \lim_{x\to 0} xq(x) = \lim_{x\to 0} -x^2 = 0$$

The auxiliary equation is $r(r-1) - r = r^2 - 2r = r(r-2)$. The exponents are $r_1 = 0$ and $r_2 = 2$.

Let $w = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}$. Then

$$\begin{aligned}
xw &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
w' &= \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} \\
xw'' &= x\sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^{n+1}
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= xw'' - w' - xw = \sum_{n=1}^{\infty} (n+1) n a_{n-1} x^n - \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=3}^{\infty} a_{n-3} x^n \\
&= 3a_1 x + \sum_{n=3}^{\infty} \left((n+1) n a_{n-1} - (n+1) a_{n-1} - a_{n-3}\right) x^n
\end{aligned}$$

This gives $a_1 = 0$ and

$$\begin{aligned}
(n+1)(n-1) a_{n-1} - a_{n-3} &= 0 \\
a_{n-1} &= \frac{a_{n-3}}{(n+1)(n-1)}
\end{aligned}$$

or after shifting the index up by 1

$$a_n = \frac{a_{n-2}}{(n+2)n}$$

Since $a_1 = 0$, $a_n = 0$ for all n odd. If n is even, let $n = 2k$. Then

$$\begin{aligned}
a_{2k} &= \frac{a_{2k-2}}{(2k+2)(2k)} = \frac{a_{2k-4}}{(2k+2)(2k)(2k)(2k-2)} = \cdots = \frac{a_0}{(2k+2)(2k)^2(2k-2)^2\cdots 4^2\cdot 2} \\
&= \frac{2(2k+2)a_0}{(2k+2)^2(2k)^2(2k-2)^2\cdots 4^2\cdot 2^2} = \frac{2(2k+2)a_0}{((2k+2)(2k)\cdots 4\cdot 2)^2} \\
&= \frac{2(2k+2)a_0}{(2(k+1)2(k)\cdots 2)^2} = \frac{2(2k+2)a_0}{(2(k+1)2(k)\cdots 2)^2} = \frac{2(2k+2)a_0}{(2^{k+1}(k+1)!)^2} \\
&= \frac{(k+1)a_0}{2^{2k}(k+1)!(k+1)!} = \frac{a_0}{2^{2k}(k+1)!k!}
\end{aligned}$$

Hence the solution is

$$w(x) = a_0 x^2 \sum_{k=0}^{\infty} \frac{1}{2^{2k}(k+1)!k!} x^{2k}$$

8.6.46: Use $r = -2$ in (59) and (60) to obtain

$$\begin{aligned}
-a_1 &= 0 \\
(k-1)(k+1)a_{k+1} - a_{k-1} &= 0
\end{aligned}$$

Hence $a_1 = 0$ and after shifting the index down by 1

$$k(k-2)a_k - a_{k-2} = 0$$

For $k = 2$, this gives $-a_0 = 0$, and for $k > 2$

$$a_k = \frac{a_{k-2}}{k(k-2)}$$

We see immediately that $a_k = 0$ for all k odd. If k is even, let $k = 2j$. Then an argument just like in the previous problem shows

$$a_{2j} = \frac{a_2}{(2j)(2j-2)^2(2j-4)^2\cdots 4^2\cdot 2} = \frac{2(2j)a_2}{(2^j j!)^2} = \frac{a_2}{2^{2j-2}j!(j-1)!}$$

(Note that we can only step down to a_2, because the recurrence only works for $k > 2$.) Hence

$$y(x) = x^{-2}\sum_{j=1}^{\infty} \frac{a_2}{2^{2j-2}j!(j-1)!}x^{2j} = \sum_{j=1}^{\infty} \frac{a_2}{2^{2j-2}j!(j-1)!}x^{2j-2} = a_2 \sum_{j=0}^{\infty} \frac{1}{2^{2j}(j+1)!j!}x^{2j}$$

which is indeed a constant multiple (by a_2/a_0) of (62).