Math 21D solution Set 9

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8.9.25: Use

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!,\ (1+frac12+n)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

We will use the recursion $,(x+1) = x,(x)$ to compute

$$\begin{aligned}
,\left(n+\frac{3}{2}\right) &= \left(n+\frac{1}{2}\right),\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right),\left(n-\frac{1}{2}\right) = \dots \\
&= \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{1}{2},\left(\frac{1}{2}\right) = \frac{2n+1}{2}\frac{2n-1}{2}\cdots\frac{1}{2},\left(\frac{1}{2}\right) \\
&= \frac{(2n+1)!}{2^{n+1}(2n)(2n-2)\cdots 2},\left(\frac{1}{2}\right) \\
&= \frac{(2n+1)!}{2^{n+1}2^n n!},\left(\frac{1}{2}\right)
\end{aligned}$$

To compute $,\left(\frac{1}{2}\right)$, use substitution ($u = x^2$) and a trick involving polar coordinates you saw in integral calculus:

$$\begin{aligned}
,\left(\frac{1}{2}\right) &= \int_0^{\infty} e^{-u}u^{-1/2}du = 2\int_0^{\infty} e^{-x^2}dx \\
\left(\int_0^{\infty} e^{-x^2}dx\right)^2 &= \left(\int_0^{\infty} e^{-x^2}dx\right)\left(\int_0^{\infty} e^{-y^2}dy\right) = \int_0^{\infty}\left(\int_0^{\infty} e^{-x^2}dx\right)e^{-y^2}dy \\
&= \int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \\
&= \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta = \int_0^{\pi/2} \left.\frac{e^{-r^2}}{-2}\right|_0^{\infty} d\theta = \frac{\pi}{4}
\end{aligned}$$

Hence $,\left(\frac{1}{2}\right) = \sqrt{\pi}$, and

$$\begin{aligned}
J_{1/2}(x) &= \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^n}{(2n+1)!\sqrt{\pi}}\left(\frac{x}{2}\right)^{2n+\frac{1}{2}} = \left(\frac{x}{2}\right)^{-1/2}\sum_{n=0}^{\infty}\frac{2^{2n+1}(-1)^n}{(2n+1)!\sqrt{\pi}}\frac{x^{2n+1}}{2^{2n+1}} \\
&= \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sqrt{\frac{2}{\pi x}}\sin x
\end{aligned}$$

Now use equation (31) from the book:

$$\begin{aligned} x^{1/2}J_{-1/2}(x) &= \frac{d}{dx}(x^{1/2}J_{1/2}(x)) = \frac{d}{dx}\left(\sqrt{\frac{2}{\pi}}\sin x\right) = \sqrt{\frac{2}{\pi}}\cos x \\ J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}}\cos x \end{aligned}$$

8.9.26: Notice that repeated use of equation (33) allows us to compute $J_\nu(x)$ for $\nu = n + 1/2$. We can use $J_{1/2}(x)$ and $J_{-1/2}(x)$ to start the recursion. Since these only contain $\sin x$, $\cos x$ and $x^{-1/2}$, and equation (33) only has x in addition to these, any $J_\nu(x)$ will also contain only such terms.

Let $\nu = -1/2$:

$$J_{-3/2}(x) = \frac{2(-1/2)}{x}J_{-1/2}(x) - J_{1/2}(x) = -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

We need to do the recursion twice to find $J_{5/2}(x)$:

$$\begin{aligned} J_{5/2}(x) &= \frac{2(3/2)}{x}J_{3/2}(x) - J_{1/2}(x) = \frac{3}{x}\left(\frac{2(1/2)}{x}J_{1/2}(x) - J_{-1/2}(x)\right) - \sqrt{\frac{2}{\pi x}}\sin x \\ &= \frac{3}{x}\left(\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x\right) - \sqrt{\frac{2}{\pi x}}\sin x \\ &= \left(\frac{3}{x^2} - 1\right)\sqrt{\frac{2}{\pi x}}\sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}}\cos x \end{aligned}$$

7.2.11:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}f(t)dt = \int_0^\pi e^{-st}\sin t dt = \frac{e^{-st}}{-s}\sin t\Big|_0^\pi + \int_0^\infty \frac{e^{-st}}{s}\cos t dt \\ &= \frac{e^{-st}}{-s^2}\cos t\Big|_0^\pi - \int_0^\pi e^{-st}\sin t dt = \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} - \frac{1}{s^2}F(s) \\ \left(1 + \frac{1}{s^2}\right)F(s) &= \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} \\ F(s) &= \frac{1 + e^{-\pi s}}{1 + s^2} \end{aligned}$$

7.2.20:

$$\mathcal{L}(e^{-2t}\cos\sqrt{3}t - t^2e^{-2t}) = \frac{\mathrm{s} + 2}{(\mathrm{s} + 2)^2 + 3} - 2\ (\mathrm{s} + 2)^{-3}$$

7.2.31: a.

$$\begin{aligned} \mathcal{L}(e^{(a+ib)t})(s) &= \int_0^\infty e^{-st}e^{(a+ib)t}dt = \int_0^\infty e^{-st+(a+ib)t}dt = \frac{e^{(a-s+ib)t}}{e^{a-s+ib}}\Big|_0^\infty \\ &= -\frac{1}{a - s + ib} = \frac{1}{s - (a + ib)} \end{aligned}$$

Note: the above computation is somewhat unjustified, as you are integrating a complex-valued function as if it were real-valued. In fact, this turns out to work, but you don't officially know this until you have studied complex analysis.

b.

$$\frac{1}{s-(a+ib)} = \frac{1}{s-a-ib}\frac{s-a+ib}{s-a+ib} = \frac{s-a+ib}{(s-a)^2-(ib)^2} = \frac{s-a+ib}{(s-a)^2+b^2}$$

c.

$$\begin{aligned}
\mathcal{L}(e^{(a+ib)t})(s) &= \mathcal{L}(e^{at}\cos bt + ie^{at}\sin bt)(s) = \mathcal{L}(e^{at}\cos bt)(s) + i\mathcal{L}(e^{at}\sin bt)(s) \\
\frac{s-a+ib}{(s-a)^2+b^2} &= \frac{s-a}{(s-a)^2+b^2} + \frac{ib}{(s-a)^2+b^2} \\
\mathcal{L}(e^{at}\cos bt)(s) &= \frac{s-a}{(s-a)^2+b^2} \\
\mathcal{L}(e^{at}\sin bt)(s) &= \frac{b}{(s-a)^2+b^2}
\end{aligned}$$

7.3.24: a. Use $\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$, and the equation (1):

$$\mathcal{L}(e^{at}t^n)(s) = \frac{n!}{(s-a)^{n+1}}$$

b. Using $\mathcal{L}(t^n f(t))(s) = (-1)^n F^{(n)}(s)$ and $\mathcal{L}(e^{at})(s) = 1/(s-a)$,

$$\mathcal{L}(e^{at}t^n)(s) = (-1)^n \frac{d^n}{ds^n}\frac{1}{s-a} = (-1)^n(-1)(-2)\cdots(-n)\frac{1}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

7.3.31:

$$\mathcal{L}(g)(s) = \int_0^\infty e^{-st}g(t)dt = \int_0^c e^{-st}0dt + \int_c^\infty e^{-st}f(t-c)dt$$

Sunstitute $u = t-c$. Then $t = u+c$, and $dt = du$:

$$\int_c^\infty e^{-st}f(t-c)dt = \int_0^\infty e^{-s(u+c)}f(u)du = \int_0^\infty e^{-su}e^{-sc}f(u)du = e^{-sc}\mathcal{L}(f)(s)$$

7.3.32:

$$\mathcal{L}(g)(s) = e^{-2s}\mathcal{L}(1)(s) = \frac{e^{-2s}}{s}$$

7.4.21:

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{3}\frac{1}{s} + \frac{1}{s-1} + \frac{14}{3}\frac{1}{s-6}\right) = \frac{1}{3} + e^t + \frac{14}{3}e^{6t}$$

7.4.28:

$$\begin{aligned}
F(s) &= \frac{s^4+4}{(s^2+s)(s^2+s-6)} = 1 - \frac{2}{3}\frac{1}{s} + \frac{5}{6}\frac{1}{s+1} - \frac{17}{6}\frac{1}{s+3} + \frac{2}{3}\frac{1}{s-2} \\
\mathcal{L}^{-1}(F) &= \delta(t) - \frac{2}{3} + \frac{5}{6}e^{-t} - \frac{17}{6}e^{-3t} + \frac{2}{3}e^{2t}
\end{aligned}$$

where $\delta(t)$ is the Dirac delta function. If you don't know what that is, don't worry. I am convinced this was not meant to be the result, and the problem is wrong.

7.4.34: Notice that

$$\frac{d}{ds}\ln\frac{s-4}{s-3} = \frac{1}{s-4} - \frac{1}{s-3}$$

Now use the equation given in the problem:

$$\begin{aligned} -tf(t) &= \mathcal{L}^{-1}\left(\frac{d}{ds}\ln\frac{s-4}{s-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-4} - \frac{1}{s-3}\right) = e^{4t} - e^{3t} \\ \mathcal{L}^{-1}\left(\ln\frac{s-4}{s-3}\right) &= f(t) = -\frac{e^{4t}}{t} + \frac{e^{3t}}{t} \end{aligned}$$

7.5.12: First, let $v(t) = w(t-1)$. Then $v(0) = 3$ and $v'(0) = 7$, and

$$\begin{aligned} w''(t-1) - 2w'(t-1) + w(t-1) &= 6(t-1) - 2 \\ v''(t) - 2v'(t) + v(t) &= 6t - 8 \\ \mathcal{L}(v''(t) - 2v'(t) + v(t)) &= \mathcal{L}(6t-8) \\ s^2\mathcal{L}(v) - sv(0) - v'(0) - 2(s\mathcal{L}(v) - v(0)) + \mathcal{L}(v) &= \frac{6}{s^2} - \frac{8}{s} \\ (s^2 - 2s + 1)\mathcal{L}(v) - 3s - 1 &= \frac{6}{s^2} - \frac{8}{s} \\ \mathcal{L}(v) &= \frac{6 - 8s + 3s^3 + s^2}{s^2(s-1)^2} \\ \mathcal{L}(v) &= \frac{4}{s} + \frac{6}{s^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2} \\ v(t) &= 4 + 6t - e^t + 2te^t \end{aligned}$$

Finally,

$$w(t) = v(t+1) = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1} = 10 + 6t + e^{t+1} + 2te^{t+1}$$

7.6.6:

$$\begin{aligned} g(t) &= (t+1)u(t-2) = (3 + (t-2))u(t-2) \\ \mathcal{L}(g(t))(s) &= 3\,\frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} \end{aligned}$$

where we used $\mathcal{L}(f(t-2)u(t-2)) = e^{-2s}F(s)$ with $f(t) = 3 + t$.

7.6.25: Use Theorem 9 with period $T = 2a$:

$$\mathcal{L}(f) = \frac{\int_0^{2a} e^{-st} f(t)dt}{1 - e^{-2as}}$$

Notice that $f(t) = 0$ for $a < t < 2a$ and $f(t) = 1$ for $0 < t < a$. Hence

$$\begin{aligned} \int_0^{2a} e^{-st} f(t)dt &= \int_0^a e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^a = \frac{1 - e^{-as}}{s} \\ \mathcal{L}(f) = \frac{1 - e^{-as}}{s(1 - e^{-2as})} &= \frac{1}{s(1 + e^{-as})} \end{aligned}$$

7.6.39: First write $g(t)$ using unit step functions, and compute $\mathcal{L}(g)$:

$$\begin{aligned} g(t) &= tu(t-1)+(1-t)u(t-5) = ((t-1)+1)u(t-1)+(-4-(t-5))u(t-5) \\ \mathcal{L}(g) &= e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)+e^{-5s}\left(-\frac{4}{s}-\frac{1}{s^2}\right) \end{aligned}$$

Hence

$$\begin{aligned} y''+5y'+6y &= g(t) \\ \mathcal{L}(y''+5y'+6y) &= \mathcal{L}(g) \\ s^2\mathcal{L}(y)-sy(0)-y'(0)+5(s\mathcal{L}(y)-y(0))+6\mathcal{L}(y) &= \frac{e^{-s}}{s^2}+\frac{e^{-s}}{s}-4\frac{e^{-5s}}{s^2}-\frac{e^{-5s}}{s^2} \\ (s^2+5s+6)\mathcal{L}(y) &= \frac{e^{-s}}{s^2}+\frac{e^{-s}}{s}-4\frac{e^{-5s}}{s^2}-\frac{e^{-5s}}{s^2}+2 \end{aligned}$$

Solve for $\mathcal{L}(y)$:

$$\mathcal{L}(y) = \frac{e^{-s}-4e^{-5s}}{s^2(s+2)(s+3)}+\frac{e^{-s}-e^{-5s}}{s(s+2)(s+3)}+\frac{2}{(s+2)(s+3)}$$

To find $y(t)$, take the inverse Laplace transform using the shift formula $\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\,u(t-a)$:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left((e^{-s}-4e^{-5s})\left(\frac{1}{6}\frac{1}{s^2}-\frac{5}{36}\frac{1}{s}-\frac{1}{9}\frac{1}{s+3}+\frac{1}{4}\frac{1}{s+2}\right)\right) \\ &\quad +\mathcal{L}^{-1}\left((e^{-s}-e^{-5s})\left(\frac{1}{6}\frac{1}{s}+\frac{1}{3}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+2}\right)+\frac{2}{s+2}-\frac{2}{s+3}\right) \\ &= \left(\frac{t-1}{6}-\frac{5}{36}-\frac{e^{-3(t-1)}}{9}+\frac{e^{-2(t-1)}}{4}\right)u(t-1) \\ &\quad -4\left(\frac{t-5}{6}-\frac{5}{36}-\frac{e^{-3(t-5)}}{9}+\frac{e^{-2(t-5)}}{4}\right)u(t-5) \\ &\quad +\left(\frac{1}{6}+\frac{e^{-3(t-1)}}{3}-\frac{e^{-2(t-1)}}{2}\right)u(t-1) \\ &\quad -\left(\frac{1}{6}+\frac{e^{-3(t-5)}}{3}-\frac{e^{-2(t-5)}}{2}\right)u(t-5)+2e^{-2t}-2e^{-3t} \\ &= \left(-\frac{5}{36}+\frac{t}{6}-\frac{1}{4}e^{-2(t-1)}+\frac{2}{9}e^{-3(t-1)}\right)u(t-1) \\ &\quad +\left(\frac{11}{36}-\frac{t}{6}+\frac{7}{4}e^{-2(t-5)}-\frac{11}{9}e^{-3(t-5)}\right)u(t-5)-2e^{-3t}+2e^{-2t} \end{aligned}$$

7.9.2: Take the Laplace transform:

$$\begin{aligned} s\mathcal{L}(x)+1 &= \mathcal{L}(x)-\mathcal{L}(y) \\ s\mathcal{L}(y) &= 2\mathcal{L}(x)+4\mathcal{L}(y) \end{aligned}$$

and solve these linear equations:

$$\begin{aligned} L(x) &= -\frac{s-4}{s^2-5\,s+6} = \frac{-2}{s-2} + \frac{1}{s-3} \\ L(y) &= -\frac{2}{s^2-5\,s+6} = \frac{2}{s-2} - \frac{2}{s-3} \end{aligned}$$

Now take the inverse Laplace transform:

$$\begin{aligned} x(t) &= e^{3\,t} - 2e^{2\,t} \\ y(t) &= 2\,e^{2\,t} - 2\,e^{3\,t} \end{aligned}$$

7.9.11: Take the Laplace transform:

$$\begin{aligned} s\mathcal{L}(x) + \mathcal{L}(y) &= \frac{1}{s} - \frac{e^{-2s}}{s} \\ \mathcal{L}(x) + s\mathcal{L}(y) &= 0 \end{aligned}$$

and solve these linear equations:

$$\begin{aligned} \mathcal{L}(x) &= \frac{1-e^{-2s}}{s^2-1} = (1-e^{-2s})\frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \\ \mathcal{L}(y) &= \frac{e^{-2s}-1}{s(s^2-1)} = (e^{-2s}-1)\frac{1}{2}\left(\frac{1}{s-1} - \frac{2}{s} + \frac{1}{s+1}\right) \end{aligned}$$

Now take the inverse Laplace transform:

$$\begin{aligned} x(t) &= \frac{e^{t}-e^{-t}}{2} - \frac{e^{t-2}-e^{-(t-2)}}{2}u(t-2) \\ &= \sinh(t) - \sinh(t-2)u(t-2) \\ y(t) &= \frac{e^{t}-e^{-t}-2}{2} - \frac{e^{t-2}-e^{-(t-2)}-2}{2}u(t-2) \\ &= \sinh(t) - 1 - (\sinh(t-2)-1)u(t-2) \end{aligned}$$