Math 244 Exam 1 Solutions

Sep 29, 2014

1. (10 pts) Explain what is wrong with the following statement: a function $f(x, y)$ with linear cross-sections for x fixed and linear cross-sections for y fixed is a linear function.

   The problem is that such a function need not be linear. For a counterexample, consider $f(x, y) = xy$. If we fix $x = k$, we get $z = ky$, which is the equation of a line in the yz-plane or in the plane $x = k$, which is parallel to the yz-plane. Similarly, fixing $y = m$ gives $z = mx$, which is the equation of a line in the plane $y = m$. Hence all of the cross-sections of $f(x, y)$ with x or y fixed are linear, but $f(x, y) = xy$ is not a linear function.

   Note: The statement does not say anything about fixing both x and y at the same time. Also, it is not true that $f(x, y)$ cannot be a linear function. For sure, if $f(x, y)$ is a linear function, its graph is a plane and its cross-sections with x or y fixed are lines, if they exist.

2. (10 pts) Show that if $\vec{u}$ and $\vec{v}$ are two vectors such that $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$ for every vector $\vec{w}$, then $\vec{u} = \vec{v}$. (Hint: no, you can't just divide both sides by $\vec{w}$.)

   First, let me try to lay a few misconceptions to rest:

   Canceling $\vec{w}$: If u, v, w are numbers and $w \neq 0$, then if $uw = vw$, you can cancel w–i.e. divide both sides by w–to get $u = v$. It is tempting to do this with $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$. But it does not work because there is no such thing as division by a vector. In fact, it is easy to find an example of three nonzero vectors that satisfy $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$. E.g. $(1, 1) \cdot (1, 0) = (1, -1) \cdot (1, 0)$.

   If two sums are equal, then they must have corresponding equal terms: If $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, and $w = (w_1, \dots, w_n)$, then

   $$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$$
   $$u_1w_1 + u_2w_2 + \dots u_nw_n = v_1w_1 + v_2w_2 + \dots v_nw_n.$$

   It is tempting to say that this implies

   $$\begin{aligned} u_1w_1 &= v_1w_1 \\ u_2w_2 &= v_2w_2 \\ &\vdots \\ u_nw_n &= v_nw_n \end{aligned}$$

   but that is false. If you want a counterexample, try $1 + 2 + 3 = 4 + 1 + 1$. BTW, even if you know $u_iw_i = v_iw_i$, you can cancel w_i only if $w_i \neq 0$.

   If two products are equal, then they must have corresponding equal factors: This is a variant on the previous theme. Let ϕ be the angle between $\vec{u}$ and $\vec{w}$ and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then

   $$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} \implies |\vec{u}||\vec{w}|\cos(\phi) = |\vec{v}||\vec{w}|\cos(\theta) \implies |\vec{u}|\cos(\phi) = |\vec{v}|\cos(\theta)$$

   But this does not show that $|\vec{u}| = |\vec{v}|$ and $\cos(\phi) = \cos(\theta)$. Counterexample: $2 \cdot 6 = 3 \cdot 4$. BTW, even if you know that $\phi = \theta$, that would not tell you $\vec{u}$ and $\vec{v}$ point in the same direction. Counterexample: the angle between $\vec{u} = (1, 0)$ and $\vec{w} = (1, 1)$ and the angle between $\vec{v} = (0, 1)$ and $\vec{w}$ are both $45°$.

   Assuming that the angles between $\vec{u}$ and $\vec{w}$ and between $\vec{v}$ and $\vec{w}$ are the same: If you write $\vec{u} \cdot \vec{w} = |\vec{u}||\vec{w}|\cos(\phi)$ and $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\phi)$, then you are assuming that theangles between $\vec{u}$ and $\vec{w}$ and between $\vec{v}$ and $\vec{w}$ are the same. But a priori there is no reason to believe that.

So, I will give you two correct arguments why $\vec{u}\cdot\vec{w}=\vec{v}\cdot\vec{w}$ for every vector $\vec{w}$ implies $\vec{u}=\vec{v}$. First, let $\vec{u}=(x_1,\ldots,x_n)$ and $\vec{v}=(y_1,\ldots,y_n)$. Let $\vec{w_i}$ be the vector all of whose coordinates are 0 except for the i-th coordinate, which is 1. Since $\vec{u}\cdot\vec{w}=\vec{v}\cdot\vec{w}$ for every $\vec{w}$, in particular, $\vec{u}\cdot\vec{w_i}=\vec{v}\cdot\vec{w_i}$ for $i=1,\ldots,n$. But $\vec{u}\cdot\vec{w_i}=x_i$ and $\vec{v}\cdot\vec{w_i}=y_i$, so $x_i=y_i$ for $i=1,\ldots,n$. Now we can conclude $\vec{u}=\vec{v}$.

Here is the other argument:

$$\vec{u}\cdot\vec{w}=\vec{v}\cdot\vec{w} \implies \vec{u}\cdot\vec{w}-\vec{v}\cdot\vec{w}=0 \implies (\vec{u}-\vec{v})\cdot\vec{w}=0.$$

This is true for every vector $\vec{w}$, so it must be true for $\vec{w}=\vec{u}-\vec{v}$. Hence

$$0=(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})=|\vec{u}-\vec{v}|^2 \implies |\vec{u}-\vec{v}|=0 \implies \vec{u}-\vec{v}=0 \implies \vec{u}=\vec{v}.$$

3. (10 pts) Let

$$f(x,y)=\begin{cases}\frac{2x^2-y^2}{xy} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Is f continuous at $(x,y)=(0,0)$. (Hint: consider $\lim_{(x,y)\to(0,0)} f(x,y)$ along lines $y=mx$ for different values m.)

This is just like the example we did in class with a slightly different function. Or compare it with 12.6.16 on your offline homework or problems 10-12 in Chapter12Section4-6 in Webwork.

We will let $(x,y)\to(0,0)$ along the lines $y=mx$. Then

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} f(x,mx) &= \lim_{(x,mx)\to(0,0)} \frac{2x^2-(mx)^2}{x(mx)}\\ &= \lim_{(x,mx)\to(0,0)} \frac{(2-m)x^2}{mx^2}\\ &= \lim_{(x,mx)\to(0,0)} \frac{2-m}{m}\\ &= \frac{2-m}{m}\end{aligned}$$

where we could cancel x^2 because $x\neq 0$ as $x\to 0$. Notice that $(2-m)/m$ has different values for different values of m. Hence $f(x,y)$ approaches different numbers as $(x,y)\to(0,0)$ depending on the path that is followed. Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence $f(x,y)$ is not continuous at $(0,0)$.

4. (10 pts) One sunny afternoon, you decide to go for a flight on Pete, your pet pterodactyl. Your destination is 120 miles to the northeast (45° east of north). It is well know that a pterodactyl's cruising airspeed is 60 miles/hour. The wind is blowing from the west at 15 miles/hour. What heading on the compass should you fly to arrive at your destination? How long will he take to get there?

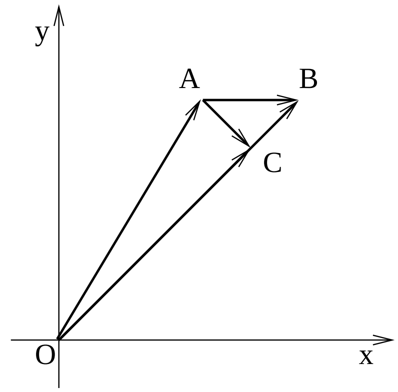
You can do this exactly like the airplane example in class, or whatever way you used to do 13.2.18 on the offline homework or problem 19 in Chapter 13 in Webwork.

I will use parallel and perpendicular components to solve this problem.

Pete's intended direction of travel is to the northeast, i.e. in the direction of the vector $(1,1)$. So let $\vec{u}=(1/\sqrt{2},1/\sqrt{2})$ be the unit vector pointing in that direction. Let $\vec{w}=(15,0)$ be the velocity of the wind, let $\vec{a}$ be Pete's velocity relative to air, and let g be Pete's velocity relative to the ground.

To make the diagram to the right less crowded, I did not label all of these vectors. Here is what they are:

$$\begin{array}{ll} \vec{a} = \vec{OA} & \vec{w} = \vec{AB} \\ \vec{a}_{||} = \vec{CA} & \vec{w}_{||} = \vec{CB} \\ \vec{a}_{\perp} = \vec{CA} & \vec{w}_{\perp} = \vec{AC} \\ \vec{g} = \vec{OB} & \end{array}$$



Note that $\vec{u}$ is not shown in the diagram, but it's just a unit vector in the same direction as $\vec{g}$.

Here is what we know

$$\begin{aligned} \vec{g} &= \vec{a} + \vec{w} \\ |\vec{a}| &= 60 \text{ mph} \\ \vec{g} &= |\vec{g}|\vec{u} && \text{b/c } \vec{g} \text{ should point to the northeast} \end{aligned}$$

We will decompose $\vec{w} = \vec{w}_{\perp} + \vec{w}_{||}$ and $\vec{a} = \vec{a}_{\perp} + \vec{a}_{||}$, where $\vec{w}_{\perp}$, $\vec{w}_{||}$, $\vec{a}_{\perp}$, and $\vec{a}_{||}$ are components perpendicular and parallel to $\vec{u}$ respectively. First

$$\begin{aligned} \vec{w}_{||} &= (\vec{w} \cdot \vec{u})\vec{u} = (15, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{15}{2}, \frac{15}{2}\right) \\ \vec{w}_{\perp} &= \vec{w} - \vec{w}_{||} = \left(\frac{15}{2}, -\frac{15}{2}\right) \end{aligned}$$

Notice that $\vec{w}_{\perp}$ wants to push you off course, so you need to compensate for it by letting $\vec{a}_{\perp} = \vec{w}_{\perp}$, hence $\vec{a}_{\perp} = (-15/2, 15/2)$. So the wind correction angle ϕ satisfies

$$\sin(\phi) = \frac{|\vec{a}_{\perp}|}{\vec{a}} = \frac{\sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{15}{2}\right)^2}}{60} = \frac{1}{4\sqrt{2}} \implies \phi \approx 10.18^\circ$$

Hence Pete should be steered 10.18° toward the north, or to a heading of $45^\circ - 10.18^\circ = 34.82^\circ$ on the compass.

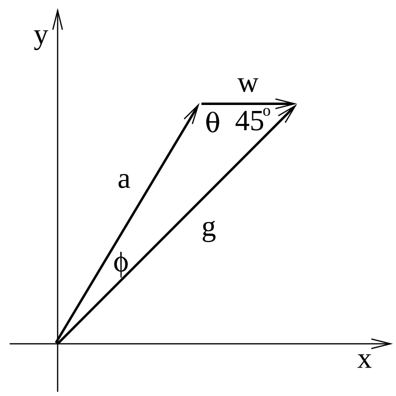
To compute Pete's ground speed, note that

$$|\vec{a}_{||}|^2 = |\vec{a}|^2 - |\vec{a}_{\perp}|^2 \implies |\vec{a}_{||}| = \sqrt{60^2 - \frac{15^2}{2}} \approx 59.06 \text{ mph.}$$

Pete is also helped by $\vec{w}_{||}$, so his ground speed will be $59.06 + 15/\sqrt{2} \approx 69.66$ mph. So Pete will take $120/69.66 \approx 1.72$ h to get to the destination.

Some of you did this problem using only high school geometry and the Law of Sines. It's a clever solution and actually a little simpler than the one above with vectors, so I would like to share it with the rest of you.

Consider the diagram below.



The triangle formed by the wind and Pete's velocities relative to the air and the ground has two sides that are known: $|\vec{w}| = 15$ and $|\vec{a}| = 60$ and one known angle of $45°$ opposite $\vec{a}$. If ϕ is the angle opposite $\vec{w}$, we can use the Law of Sines to write

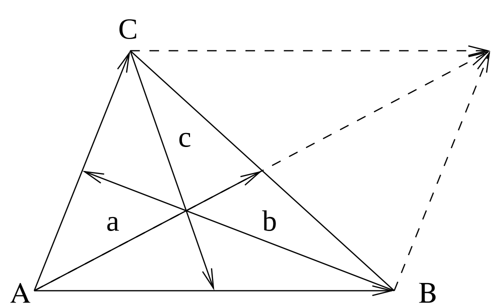
$$\frac{\sin(\phi)}{15} = \frac{\sin(45°)}{60} \implies \sin\phi = \frac{15}{60}\sin(45°) = \frac{1}{4\sqrt{2}} \implies \phi \approx 10.18°.$$

Hence Pete should follow a heading of $45° - 10.18° = 34.82°$ on the compass. To find Pete's ground speed, we note that the angle opposite $\vec{g}$ is $\theta \approx 180° - 45° - 10.18° = 124.82°$. We use the Law of Sines again to find

$$\frac{|\vec{g}|}{\sin(\theta)} = \frac{60}{\sin(45°)} \implies |\vec{g}| \approx 60\sqrt{2}\sin(124.82°) \approx 69.66.$$

Which once again tells us that Pete's flying at a ground speed of 69.66mph and he should reach his destination in $120/69.66 \approx 1.72$ h.

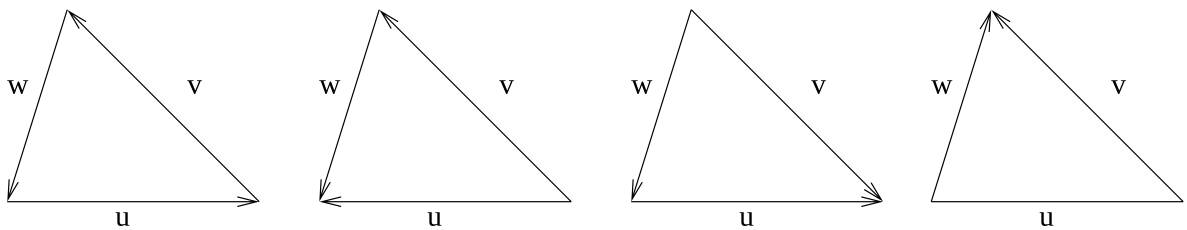
5. (10 pts) **Extra credit problem.** Let A, B, and C be the vertices of a triangle. Let $\vec{a}$ be the vector from A to the midpoint of the side BC, $\vec{b}$ the vector from B to the midpoint of the side AC, and $\vec{c}$ the vector from C to the midpoint of the side AB. Prove that there exists a triangle whose sides are $\vec{a}$, $\vec{b}$, and $\vec{c}$. (Hint: if I just gave you three random vectors, you probably couldn't make a triangle out of them. So first think about what the three vectors need to do to form the sides of a triangle.)



First notice that in any triangle ABC, $\vec{AB}+\vec{BC}+\vec{CA} = \vec{0}$. In fact, if three nonzero vectors add up to $\vec{0}$ and they are parallel, then they form a triangle. But that is only one possible scenario. We could have $\overrightarrow{BA}$ instead of $\vec{AB}$, in which case we have $-\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$. Similarly, the other two vectors could point in the opposite direction. This gives us eight possible equations, although they sort into four equivalent pairs. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three nonzero nonparallel vectors, then they form a triangle if and only if

$$\begin{aligned}
\vec{u}+\vec{v}+\vec{w}=\vec{0} &\iff -\vec{u}-\vec{v}-\vec{w}=\vec{0}\\
-\vec{u}+\vec{v}+\vec{w}=\vec{0} &\iff \vec{u}-\vec{v}-\vec{w}=\vec{0}\\
\vec{u}-\vec{v}+\vec{w}=\vec{0} &\iff -\vec{u}+\vec{v}-\vec{w}=\vec{0}\\
\vec{u}+\vec{v}-\vec{w}=\vec{0} &\iff -\vec{u}-\vec{v}+\vec{w}=\vec{0}
\end{aligned}$$

These four equations correspond to the four diagrams below:



Now, in this problem, we have $\vec{a}=(\vec{AB}+\vec{AC})/2$, $\vec{b}=(\vec{BA}+\vec{BC})/2$, $\vec{c}=(\vec{CA}+\vec{CB})/2$. It is clear from the picture that they are nonzero and cannot be parallel. Notice

$$\begin{aligned}
\vec{a}+\vec{b}+\vec{c} &= \frac{\vec{AB}+\vec{AC}}{2}+\frac{\overrightarrow{BA}+\overrightarrow{BC}}{2}+\frac{\overrightarrow{CA}+\overrightarrow{CB}}{2}\\
&= \underbrace{\frac{\overrightarrow{AB}}{2}+\frac{\overrightarrow{BA}}{2}}_{\vec{0}}+\underbrace{\frac{\overrightarrow{AC}}{2}+\frac{\overrightarrow{CA}}{2}}_{\vec{0}}+\underbrace{\frac{\overrightarrow{BC}}{2}+\frac{\overrightarrow{CB}}{2}}_{\vec{0}}\\
&= \vec{0}
\end{aligned}$$

Hence $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a triangle.