Math 244 Exam 2 Solutions

Oct 31, 2014

1. (10 pts) It can be proved that $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ for all vectors $\vec{a},\vec{b},\vec{c}\in\mathbb{R}^3$. Use this result to show that the cross product distributes over addition.

   (a) First use distributivity for the dot product to show that for any vector $\vec{d}$,

   $$[(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})]\cdot\vec{d} = [(\vec{a}+\vec{b})\times\vec{c}]\cdot\vec{d}.$$

   $$\begin{aligned}
   [(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})]\cdot\vec{d} &= (\vec{a}\times\vec{c})\cdot\vec{d} + (\vec{b}\times\vec{c})\cdot\vec{d} && \text{by distributivity of } \cdot \text{ over } + \\
   &= \vec{a}\cdot(\vec{c}\times\vec{d}) + \vec{b}\cdot(\vec{c}\times\vec{d}) && \text{by the identity in the problem} \\
   &= (\vec{a}+\vec{b})\cdot(\vec{c}\times\vec{d}) && \text{by distributivity of } \cdot \text{ over } + \\
   &= ((\vec{a}+\vec{b})\times\vec{c})\cdot\vec{d} && \text{by the identity in the problem}
   \end{aligned}$$

   (b) Now use one of the problems from your homework or the previous exam to argue that this implies

   $$(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) = (\vec{a}+\vec{b})\times\vec{c}.$$

   Since the result of part (a) holds for any vector $\vec{d}$, it follows by the result of problem 13.3.64 (or problem 2 on the first exam) that

   $$(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) = (\vec{a}+\vec{b})\times\vec{c}.$$

2. (10 pts) Let f be a differentiable function. Is it true that for any point (a, b) there is always a direction in which the rate of change of f at (a, b) is 0? Make sure you justify your answer.

   Yes, this is true. If $\operatorname{grad} f(a,b) = 0$, then any direction $\vec{u}$ will give $f_{\vec{u}}(a,b) = 0$. If $\operatorname{grad} f(a,b) \neq 0$, then let $\vec{v} = (c,d) = \operatorname{grad} f(a,b)$ and choose a unit vector $\vec{u}$ that is perpendicular to $\vec{v}$, such as

   $$\vec{u} = \frac{(-d,c)}{\sqrt{c^2+d^2}}.$$

   Since c and d are not both 0, the denominator is nonzero, so this is always a valid choice and always gives a unit vector, i.e. a direction. Now notice that

   $$f_{\vec{u}}(a,b) = \operatorname{grad} f(a,b)\cdot\vec{u} = (c,d)\frac{(-d,c)}{\sqrt{c^2+d^2}} = 0.$$

3. (10 pts) In preparation for trick-or-treating, you Google the density of candy in your neighborhood and find that it is given by the function $f(x,y) = 13\sin^2(\pi/4x + \pi/8y)$, where x and y are measured in miles, f is measured in lb/acre, and the x-axis points east and the y-axis points north. Your place is at the point $(4, 1)$ on the map. Jack O'Lantern Dr starts at your house and runs in the direction 30° east of north (i.e. 30° on the compass). Warlock Ave, also starts at your house and runs 30° south of east (i.e. 120° on the compass).

   (a) Which street should you take if you would like to head out in the direction of faster growth for candy density?

   To choose the direction of faster growth, we need to compute the directional derivatives of $f(x,y)$ in the direction of the two streets. Since Jack O'Lantern Dr runs 60° north of east, its direction vector is $\vec{u} = (\cos(60°), \sin(60°)) = (1/2, \sqrt{3}/2)$. The direction vector of Warlock Ave is $\vec{v} = (\cos(-30°), \sin(-30°)) = (\sqrt{3}/2, -1/2)$.

$$\begin{aligned} f_x(4,1) &= 26\sin\left(\frac{\pi}{4}x + \frac{\pi}{8}y\right)\cos\left(\frac{\pi}{4}x + \frac{\pi}{8}y\right)\frac{\pi}{4}\bigg|_{(4,1)} \\ &= \frac{13\pi}{2}\sin\left(\frac{\pi}{2}x + \frac{\pi}{4}y\right)\bigg|_{(4,1)} \\ &= \frac{13\pi}{2}\sin\left(2\pi + \frac{\pi}{4}\right) \\ &= \frac{13\pi\sqrt{2}}{4} \end{aligned}$$

where we used the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to go from the first line to the second line.

Similarly,

$$\begin{aligned} f_y(4,1) &= 26\sin\left(\frac{\pi}{4}x + \frac{\pi}{8}y\right)\cos\left(\frac{\pi}{4}x + \frac{\pi}{8}y\right)\frac{\pi}{8}\bigg|_{(4,1)} \\ &= \frac{13\pi}{4}\sin\left(\frac{\pi}{2}x + \frac{\pi}{4}y\right)\bigg|_{(4,1)} \\ &= \frac{13\pi}{4}\sin\left(2\pi + \frac{\pi}{4}\right) \\ &= \frac{13\pi\sqrt{2}}{8} \end{aligned}$$

Hence

$$\operatorname{grad} f(4,1) = \frac{13\pi\sqrt{2}}{8}(2,1)$$

and the directional derivatives are

$$\begin{aligned} f_{\vec{u}}(4,1) &= \frac{13\pi\sqrt{2}}{8}(2,1)\cdot\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ &= \frac{13\pi\sqrt{2}}{16}(2+\sqrt{3}) \end{aligned}$$

and

$$\begin{aligned} f_{\vec{v}}(4,1) &= \frac{13\pi\sqrt{2}}{8}(2,1)\cdot\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \\ &= \frac{13\pi\sqrt{2}}{16}(2\sqrt{3}-1) \end{aligned}$$

Since $2+\sqrt{3} > 2\sqrt{3} > 2\sqrt{3}-1$, $f_{\vec{u}}(4,1) > f_{\vec{v}}(4,1)$. Therefore Jack O'Lantern Dr is the right choice.

(b) If you were flying Pete, your pet pterodactyl, what direction would you want to fly to follow the fastest rate of growth of candy density?

Pete can fly in any direction, so you can just follow the direction of fastest growth, which is the direction of the gradient, i.e.

$$\frac{\operatorname{grad} f(4,1)}{|\operatorname{grad} f(4,1)|} = \frac{(2,1)}{\sqrt{5}}.$$

4. Let $f(t)$ be a single variable function that is strictly increasing. Let $g(x,y) = f(\sqrt{x^2+y^2})$.
   (a) Use the chain rule to find the gradient of g.

   Let $t = \sqrt{x^2+y^2}$. Then

$$\begin{aligned}
\operatorname{grad} g &= \operatorname{grad}(f(t)) \\
&= \left(\frac{df}{dt}\frac{\partial t}{\partial x}, \frac{df}{dt}\frac{\partial t}{\partial y}\right) \\
&= \left(f'(\sqrt{x^2+y^2})\frac{1}{2\sqrt{x^2+y^2}}(2x), f'(\sqrt{x^2+y^2})\frac{1}{2\sqrt{x^2+y^2}}(2y)\right) \\
&= f'(\sqrt{x^2+y^2})\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)
\end{aligned}$$

   (b) Describe the contours of g.

   The contours of g are given by the equations $c = g(x,y) = f(\sqrt{x^2+y^2})$ where c is the level. Since f is strictly increasing, for any value of c, it can have at most one value t such that $f(t) = c$. That is f is a one-to-one function. If such a t exists, then the contour corresponding to c is the curve $t = \sqrt{x^2+y^2}$. If $t \geq 0$ then, this curve is a circle of radius t centered at the origin. So the contours of g are concentric circles centered at the origin.

   Note: Some of you said that the graph of g was an upside down cone whose tip was at the origin, therefore the contours are concentric circles centered at the origin. While it is true that the contours are concentric circles centered at the origin, this reasoning is false. In general, there is no reason to believe that the graph of g is such a cone. That is only true in the very special case when $f(t) = t$. Try plotting the graph of g in Wolfram Alpha for $f(t) = t^3 + 5$ or $f(t) = e^t$ to see what kinds of surfaces you get.

5. (10 pts) **Extra credit problem.** Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. (Hint: since the function is not defined at $(0,0)$ by the same formula as everywhere else, you cannot use this formula to find its partial derivatives there. You will have to use the definition of the partial derivative.)

   To find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y) \neq (0,0)$, we can just use the quotient rule to differentiate $\frac{xy(x^2-y^2)}{x^2+y^2} = \frac{x^3y-xy^3}{x^2+y^2}$ to get

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{x^3y-xy^3}{x^2+y^2} &= \frac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2} \\
&= \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}
\end{aligned}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$
$$= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$

and

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{x^3y - xy^3}{x^2+y^2} &= \frac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)(2y)}{(x^2+y^2)^2} \\
&= \frac{x^5 + x^3y^2 - 3x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4}{(x^2+y^2)^2} \\
&= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \\
&= \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2}
\end{aligned}$$

Since $f(x,y)$ is not defined by the same formula at $(0,0)$, we must use the definition of the derivative to find its partial derivatives there:

$$\begin{aligned}
f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} = 0
\end{aligned}$$

and

$$\begin{aligned}
f_y(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} = 0
\end{aligned}$$

Hence

$$f_x(x,y) = \begin{cases} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

and

$$f_y(x,y) = \begin{cases} \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(b) Now find the mixed second partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ at $(0,0)$. Are they equal? (Hint: once again, you will have to use the definition of the derivative.)

$$\begin{aligned}
f_{xy}(0,0) &= \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{h(0^4+40^2h^2-h^4)}{(0^2+h^2)^2} - 0}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h^5}{h^4} - 0}{h}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h \to 0} \frac{-h^5}{h^5} \\
&= \lim_{h \to 0} -1 \\
&= -1
\end{aligned}
$$

and

$$
\begin{aligned}
f_{yx}(0,0) &= \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \\
&= \lim_{h \to 0} \frac{\frac{h(h^4 - 4h^2 0^2 - 0^4)}{(h^2+0^2)^2} - 0}{h} \\
&= \lim_{h \to 0} \frac{\frac{h^5}{h^4} - 0}{h} \\
&= \lim_{h \to 0} \frac{h^5}{h^5} \\
&= \lim_{h \to 0} 1 \\
&= 1
\end{aligned}
$$

So $f_{xy}(0,0) \neq f_{yx}(0,0)$ in this case.