Math 244 Final Exam Solutions

Dec 12, 2014

1. (10 pts) Show that the function f does not have a limit at $(0,0)$ by examining the limits of f as $(x,y) \to (0,0)$ along the curve $y = kx^2$ for different values of k:

$$f(x,y) = \frac{x^2}{x^2+y}, \qquad x^2 + y \neq 0.$$

Consider the limit as $(x,y) \to (0,0)$ along the curve $y = mkx^2$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2}} \frac{x^2}{x^2+y} = \lim_{x\to 0} \frac{x^2}{x^2+kx^2} = \lim_{x\to 0} \frac{x^2}{x^2(1+k)} = \lim_{x\to 0} \frac{1}{1+k} = \frac{1}{1+k}$$

where we could cancel x^2 because $x \neq 0$ as $x \to 0$. So the value of f approaches different numbers as $(x,y) \to (0,0)$ along different paths. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Consider the function $f(x,y) = \sqrt{|xy|}$.
   (a) (5 pts) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.

   Because of the absolute value, we need to use the definition of the derivative.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

   Similarly,

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

   (b) (10 pts) Is f differentiable at $(0,0)$? (Hint: Consider the directional derivative $f_{\vec{u}}(0,0)$ for $\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$. How does it compare with the definition of differentiability?)

   Let $\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$. We will compute $f_{\vec{u}}(0,0)$:

$$\begin{aligned} f_{\vec{u}}(0,0) &= \lim_{h\to 0} \frac{f\left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{\frac{h^2}{4}} - 0}{h} \qquad\qquad \text{since } \left|\frac{h^2}{4}\right| = \frac{h^2}{4} \\ &= \lim_{h\to 0} \frac{\frac{|h|}{2} - 0}{h} \\ &= \lim_{h\to 0} \frac{|h|}{2h} \end{aligned}$$

   This limit does not exist because the left and the right hand limits are $-1/2$ and $1/2$. This already suggests that f is not differentiable at $(0,0)$ because if it were, its directional derivative there would be $\vec{\nabla} f \cdot \vec{u} = \vec{0} \cdot \vec{u} = 0$. However, we can see directly that f fails to satisfy the definition of differentiability at $(0,0)$.
   If f is differentiable at $(0,0)$, then it must have a good linear approximation given by the partial derivatives $L(x,y) = 0(x-0) + 0(y-0) + 0 = 0$ in the sense that

$E(x,y) = f(x,y) - L(x,y) = \sqrt{|xy|}$ must satisfy

$$0 = \lim_{\substack{h\to 0\\ k\to 0}} \frac{E(h,k)}{\sqrt{h^2+k^2}}$$
$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$$

In particular, when $k = h$ as $h \to 0$, we get

$$\lim_{h\to 0} \frac{\sqrt{|h^2|}}{\sqrt{h^2+h^2}} = \lim_{h\to 0} \frac{\sqrt{h^2}}{\sqrt{2}\sqrt{h^2}} = \lim_{h\to 0} \frac{\sqrt{h^2}}{\sqrt{2}\sqrt{h^2}} = \frac{1}{\sqrt{2}} \neq 0.$$

This tells us that

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{E(h,k)}{\sqrt{h^2+k^2}}$$

cannot possibly be 0. Hence f is not differentiable at $(0,0)$.

3. For constants a and b with $ab \neq 0$ and $ab \neq 1$, let $f(x,y) = ax^2 + by^2 - 2xy - 4x - 6y$.
   (a) (8 pts) Find the x- and y-coordinates of the critical point. Your answer will be in terms of a and b.

   To find the critical point(s), we look for where the gradient is 0 or undefined. In this case

   $$\vec{\nabla} f = (2ax - 2y - 4, 2by - 2x - 6),$$

   which is defined for any x, y, and is 0 when

   $$2ax - 2y - 4 = 0$$
   $$2by - 2x - 6 = 0$$

   The first equation implies $y = ax - 2$. Substitute this into the second equation to get

   $$0 = 2b(ax-2) - 2x - 6 = 2abx - 4b - 2x - 6 = (2ab-2)x - 4b - 6 \implies$$

   $$x = \frac{2b+3}{ab-1} \implies y = a\frac{2b+3}{ab-1} - 2 = \frac{2ab+3a-2(ab-1)}{ab-1} - 2 = \frac{3a+2}{ab-1}.$$

   So the critical point is at $\left(\frac{2b+3}{ab-1}, \frac{3a+2}{ab-1}\right)$.

   (b) (7 pts) Classify the critical point for all values of a and b with $ab \neq 0$ and $ab \neq 1$.

   To classify the critical point, we will use the second derivative test and look at

   $$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (2a)(2b) - (-2)^2 = 4ab - 4.$$

   $D < 0$: If $4ab - 4 < 0$, that is $ab < 1$, then the critical point is a saddle point.

   $D = 0$: This never happens as $ab \neq 1$.

   $D > 0$: If $4ab - 4 > 0$, that is $ab > 1$, then the critical point is a local extremum. It is a local maximum if $0 > \frac{\partial^2 f}{\partial x^2} = 2a$, that is when $a < 0$. It is a local minimum if $0 < \frac{\partial^2 f}{\partial x^2} = 2a$, that is when $a > 0$. Note that $a = 0$ never happens, otherwise $ab = 0$.

4. (10 pts) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$. One of your homework problems was to show that if $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$ for every vector $\vec{w}$, then $\vec{u} = \vec{v}$. Is it true that if $\vec{u} \times \vec{w} = \vec{v} \times \vec{w}$ for every vector $\vec{w} \in \mathbb{R}^3$ then $\vec{u} = \vec{v}$? If so, prove it; if not, give a counterexample.

Yes, this is true. Notice that

$$\vec{u} \times \vec{w} = \vec{v} \times \vec{w} \implies \vec{0} = \vec{u} \times \vec{w} - \vec{v} \times \vec{w} = (\vec{u} - \vec{v}) \times \vec{w}.$$

Let $\vec{a} = \vec{u} - \vec{v}$. We will show that if $\vec{a} \times \vec{w} = \vec{0}$ for all $\vec{w}$, then $\vec{a}$ must be $\vec{0}$. Suppose this is not true and $\vec{a} \neq \vec{0}$. Choose any nonzero vector that is not parallel to $\vec{a}$. You can always find such a vector in $\mathbb{R}^3$. For example, you can take one of $\vec{i}$, $\vec{j}$ or $\vec{k}$, as $\vec{a}$ cannot be parallel to all three. Now, since $\vec{a} \times \vec{w} = \vec{0}$ for any $\vec{w}$, it must be that

$$0 = |\vec{a} \times \vec{w}| = |\vec{a}||\vec{w}| \sin(\phi)$$

where ϕ is the angle between $\vec{a}$ and $\vec{w}$. Since $\vec{a} \nparallel \vec{w}$, $\phi \neq 0, \pi$ and hence $\sin(\phi) \neq 0$. But $|\vec{a}| \neq 0$ and $|\vec{w}| \neq 0$, so the right hand side above cannot be 0. So we have reached a contradiction. Therefore the assumption that $\vec{a} \neq \vec{0}$ must have been false.

5. Santa Claus needs to buy presents for 7500 good girls and boys. He wants to give Pete, the flying pterodactyl™ action figures to some of the children and handheld Mathdoku video games to the others. It costs Santa $10000 + 100\sqrt{x}$ dollars to buy x Petes and $20000 + 5000\ln(y)$ dollars to buy y Mathdoku games from fair trade suppliers.

(a) (10 pts) Santa is a generous guy, so he actually wants to maximize his cost to benefit his suppliers. How many Petes and how many Mathdokus should he order?

Suppose Santa buys x Petes and y Mathdoku games. Then his total cost is

$$f(x, y) = 10000 + 100\sqrt{x} + 20000 + 5000\ln(y) = 30000 + 100\sqrt{x} + 5000\ln(y).$$

The constraint is $g(x, y) = x + y = 7500$. By the method of Langrange multipliers, the critical points are where $\vec{\nabla} f = \lambda \vec{\nabla} g$ for some scalar λ. So

$$\vec{\nabla} f = \left(\frac{1}{2} 100 x^{-1/2}, 5000 \frac{1}{|y|} \right) = \lambda(1, 1).$$

Since only nonnegative values of y make sense, $|y| = y$. Thus we have the system of equations

$$\begin{aligned} \frac{50}{\sqrt{x}} &= \lambda \\ \frac{5000}{y} &= \lambda \\ x + y &= 7500 \end{aligned}$$

Note $y \neq 0$ otherwise $5000/y$ cannot be equal to λ. So the second equation gives $y = 5000/\lambda$. Similarly, $x \neq 0$, so $\sqrt{x} = 50/\lambda$ and then $x = 2500/\lambda^2$. Substituting these into the third equation yields

$$\begin{aligned}
\frac{2500}{\lambda^2} + \frac{5000}{\lambda} &= 7500 \\
2500 + 5000\lambda &= 7500\lambda^2 \\
7500\lambda^2 - 5000\lambda - 2500 &= 0 \\
3\lambda^2 - 2\lambda - 1 &= 0 \\
(3\lambda + 1)(\lambda - 1) &= 0 \\
\lambda &= -\frac{1}{3} \text{ or } 1
\end{aligned}$$

If $\lambda = -1/3$, then $y < 0$, which does not make sense in the context of this problem. So $\lambda = 1$, and hence $x = 2500$ and $y = 5000$.
The maximum could occur at the critical point $(2500, 5000)$ or at the endpoints $(7500, 0)$ and $(0, 7500)$. But notice that f does not have a value at $(7500, 0)$, so the endpoint there is really $(7499, 1)$. The values of f at these points are

$$\begin{aligned}
f(2500, 5000) &= 30000 + 100\sqrt{2500} + 5000\ln(5000) \approx 77586 \\
f(7499, 1) &= 30000 + 100\sqrt{7499} + 5000\ln(1) \approx 38660 \\
f(0, 7500) &= 30000 + 100\sqrt{0} + 5000\ln(7500) \approx 74613
\end{aligned}$$

So the cost appears to be maximized if Santa buys 2500 Petes and 5000 Mathdoku games.

(b) (3 pts) How do you know that what you found in part(a) is a global maximum?

The constraint $x + y = 7500$ with the two endpoints $(0, 7500)$ and $(7499, 1)$ (since $y = 0$ is not an option) is a line segment. That is a closed and bounded set. The function f is continuous on this set. So we know f must have a global maximum ont this set, and the global maximum must be at a local maximum or at one of the endpoints. We have checked all three such points. The one with the highest value of f must be the global maximum.

(c) (2 pts) Estimate how much more it would cost to give out 7501 presents while maximizing Santa's cost.

As we showed in class, the increase in maximum cost is approximately λ, which is 1. So the cost would increase by about \$1.

6. An object shaped like an "ice cream cone" is bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and the sphere $x^2 + y^2 + z^2 = 9$. The density of its material increases with distance from the origin and is given by the function $f(x, y, z) = 3 + \sqrt{x^2 + y^2 + z^2}$ where x, y, and z are measured in cm, and f is in g/cm^3.

(a) (10 pts) Choose appropriate coordinates and find the mass of this object.

It is easiest to describe this region in terms of spherical coordinates: $0 \le \rho \le 3$, $0 \le \phi \le \pi/6$, $0 \le \theta \le 2\pi$. In terms of these, $f(\rho, \phi, \theta) = 3 + \rho$. The mass is the integral of the density:

$$\begin{aligned}
\int_R f(\rho,\phi,\theta)\,dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 (3+\rho)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \sin(\phi)\left(\rho^3+\frac{\rho^4}{4}\right)\Big|_0^3\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \sin(\phi)\left(27+\frac{81}{4}\right)\,d\phi\,d\theta \\
&= \frac{189}{4}\int_0^{2\pi} -\cos(\phi)|_0^{\pi/6}\,d\theta \\
&= \frac{189}{4}\int_0^{2\pi} 1-\frac{\sqrt{3}}{2}\,d\theta \\
&= \frac{189}{4}\left(1-\frac{\sqrt{3}}{2}\right)\int_0^{2\pi} d\theta \\
&= \frac{189}{4}\left(1-\frac{\sqrt{3}}{2}\right)2\pi \\
&= \frac{189(2-\sqrt{3})\pi}{4} \approx 39.77
\end{aligned}$$

So the mass of the object is about 39.77 g.

(b) (5 pts) Choose a coordinate system different from the one you used in part (a) and set up an integral for the mass of the object. You do not need to evaluate the integral.

To describe the region in cylindrical coordinates, note that the conical side is given by $z = \sqrt{3x^2+3y^2} = \sqrt{3r^2} = \sqrt{3}r$ and the spherical cap is given by $9 = x^2+y^2+z^2 = r^2+z^2$. These intersect where

$$r^2 + (\sqrt{3}r)^2 = 9 \implies 4r^2 = 9 \implies r^2 = \frac{9}{4} \implies r = \frac{3}{2}.$$

Hence the integral in cylindrical coordinates is

$$\int_0^{2\pi}\int_0^{\frac{3}{2}}\int_{\sqrt{3}r}^{\sqrt{9-r^2}} (3+\sqrt{r^2+z^2})r\,dz\,dr\,d\theta$$

In Cartesian coordinates, the region is described by $-\frac{3}{2} \le x \le \frac{3}{2}$, $-\sqrt{\frac{9}{4}-x^2} \le y \le \sqrt{\frac{9}{4}-x^2}$, and $\sqrt{3x^2+3y^2} \le z \le \sqrt{9-x^2-y^2}$. Hence the integral is

$$\int_{-\frac{3}{2}}^{\frac{3}{2}}\int_{-\sqrt{\frac{9}{4}-x^2}}^{\sqrt{\frac{9}{4}-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{9-x^2-y^2}} 3+\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$$

7. **Extra credit problem.** Let $f(x,y)$ be a function. Suppose that all of the second partial derivatives of f exist.

(a) (3 pts) Under what conditions would $f_{xy}(a,b) = f_{yx}(a,b)$ have to be true?

Schwarz's theorem says that if (a,b) is an interior point of the domain of f and f_{xy} and f_{yx} are both continuous at (a,b) then they must be equal there.

(b) (12 pts) Find a function f whose mixed second partial derivatives are not equal at some point (a, b) and show that $f_{xy}(a, b) \neq f_{yx}(a, b)$. (Hint: such a function may not be defined by a single formula everywhere.)

Let

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

See the solution to problem 5 on the second midterm exam for the proof that $f_{xy} \neq f_{yx}$.