Math 244 worksheet on differentiability

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First read Section 14.8–a few times if necessary–and try to understand as much of it as you can.

1. Recall from Calculus I the following definition:

   Definition 1. A function $f(x)$ is differentiable at x_0 if
   $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
   exists.

   Another way to characterize differentiability is the following definition.

   Definition 2. A function $f(x)$ is differentiable at x_0 if there is a linear function $L(x) = m(x-x_0)+b$ that is a good approximation to f in the sense that the error $E(x) = f(x)-L(x)$ satisfies
   $$\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0.$$

   The idea here is that $L(x)$ is a good approximation to $f(x)$ if the error goes to 0 as $x \to x_0$ faster than $x - x_0$ goes to 0.

   Recall that $f(x)$ is continuous at x_0 if
   $$\lim_{x \to x_0} f(x) = f(x_0).$$
   Theorem 2.1 in Section 2.6 in your textbook says that if $f(x)$ satisfies Definition 1 then $f(x)$ is continuous at x_0. Prove this (better) or review the proof in the book (not as good).

   Now prove that if $f(x)$ satisfies Definition 2 then $f(x)$ must be continuous at x_0.

2. Show that if $f(x)$ satisfies Definition 2 then $b = f(x_0)$.

3. Show that if $f(x)$ satisfies Definition 2 then
   $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
   exists and $m = f'(x_0)$.

   Compare your proof with the one given in Example 1 in Section 14.8.

4. Now, show that if $f(x)$ satisfies Definition 1 then $L(x) = f'(x_0)(x - x_0) + f(x_0)$ is a linear approximation to f such that the error $E(x) = f(x) - L(x)$ satisfies
   $$\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0.$$

   Note that this says if $f(x)$ satisfies Definition 1 then it must also satisfy Definition 2. In part 3, you proved that if $f(x)$ satisfies Definition 2 then it must also satisfy Definition 1. Therefore the two definitions are logically edquivalent.

   Definition 1 does not readily extend to the multivariable setting since a multivariable function has several partial derivatives. The definition of differentiability in Section 14.8 is the generalization of Definition 2 to the multivariable setting.

5. Prove that if $f(x, y)$ is differentiable at the point (x_0, y_0), then it must be continuous there. (Hint: try to generalize your proof from part 1 to the multivariable setting.)