Math 244 Exam 1 Solutions

Feb 18, 2015

1. (5 pts each)
   (a) Explain what is wrong with the following statement: the functions $f(x, y) = \sqrt{x^2 + y^2}$ and $g(x, y) = x^2 + y^2$ have the same contour diagram.

   In both cases, the contours are concentric circles centered at the origin. But while the circle of radius r corresponds to the level $f = r$ in the contour diagram of f, it corresponds to the level $g = r^2$ in the contour diagram of g.

   Another thing you can say is that if you consider the contour curves at positive integer levels only, the contours of f are concentric circles centered at the origin which are evenly spaced with radii $1, 2, 3, \ldots$, whereas the contours of g are concentric circles centered at the origin with radii $1, \sqrt{2}, \sqrt{3}, \ldots$, i.e. they get closer to each other farther away from the origin. But note that the levels of a function are not in general assumed to be positive integers, so you need to say that you are talking about the contours at positive integer levels only.

   (b) Is the following statement true or false? Justify your answer.
   Two isotherms representing distinct temperatures on a weather map cannot intersect.

   This is true because if they did, the temperature would have to have two different values at that point, which is obviously not possible. In fact, it is true for any function that its contours cannot intersect, otherwise the function would have to have multiple values at such points of intersection.

2. (10 pts) Show that the function f does not have a limit at $(0, 0)$ by examining the limits of f as $(x, y) \to (0, 0)$ along the curve $y = kx^2$ for different values of k:

$$f(x, y) = \frac{x^2}{x^2 + y}, \quad \text{for} \qquad x^2 + y \neq 0.$$

   Consider the limit as $(x, y) \to (0, 0)$ along the curve $y = mkx^2$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2}} \frac{x^2}{x^2 + y} = \lim_{x\to 0} \frac{x^2}{x^2 + kx^2} = \lim_{x\to 0} \frac{x^2}{x^2(1 + k)} = \lim_{x\to 0} \frac{1}{1 + k} = \frac{1}{1 + k}$$

   where we could cancel x^2 because $x \neq 0$ as $x \to 0$. So the value of f approaches different numbers as $(x, y) \to (0, 0)$ along different paths. Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

3. (5 pts each)
   (a) Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 + 4z$.

   The level surfaces are given by the equations $c = f(x, y, z) = x^2 + y^2 + 4z$ for fixed values of c. We can solve for z to get

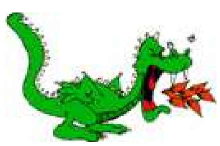
$$z = \frac{c}{4} - \frac{1}{4}(x^2 + y^2).$$

   We saw in class that the graph of $z = x^2 + y^2$ is a paraboloid. Hence the level surfaces of f are upside down paraboloids that cross the z-axis at $c/4$.

(b) Is there a function $g(x,y)$ whose graph is the same as the level surface $f = 0$? If so, find g; if not, explain why there is no such function.

$$f = 0 \implies x^2 + y^2 + 4z = 0 \implies z = -\frac{x^2 + y^2}{4}$$

Hence the graph of the function $g(x,y) = -(x^2 + y^2)/4$ is the same as the level surface $f = 0$.



4. (10 pts) One fine day, Smaug, the fearsome dragon decides to fly from his home on the Lonely Mountain to the city of Dale to terrorize the population there. Dale is 100 miles south of the Lonely Mountain. Smaug can fly at a speed of 65 mph. However fierce he is, Smaug still has to comply with Middle Earth air traffic regulations, which restrict southbound dragons to fly at an altitude of either 4500 ft or 6500 ft above sea level. At 4500 ft, the wind blows from 30° south of west at 8 mph. At 6500 ft, the wind blows from the west at 25 mph. Which altitude should Smaug choose to fly at if he wants to get to Dale as quickly as possible?

For flying at 4500 ft, refer to the figure on the right, where $\vec{v}_a$ is Smaug's air velocity, $\vec{v_g}$ is his ground velocity, and $\vec{w}$ is the wind. Since $\vec{w}$ points 30° north of east, we have

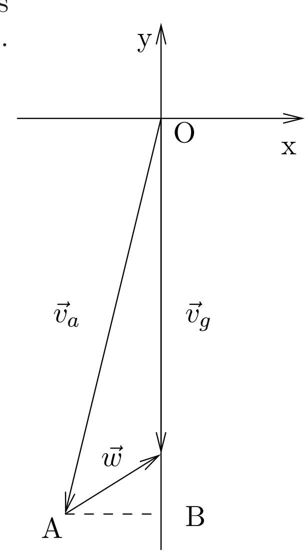
$$\vec{w} = (8\cos(30^\circ), 8\sin(30^\circ)) = (4\sqrt{3}, 4).$$

$\triangle OAB$ is a right triangle, so by the Pythagorean Theorem,

$$|\overrightarrow{OB}| = \sqrt{|\overrightarrow{OA}|^2 - |\overrightarrow{AB}|^2} = \sqrt{65^2 - (4\sqrt{3})^2} = \sqrt{4177}.$$

And

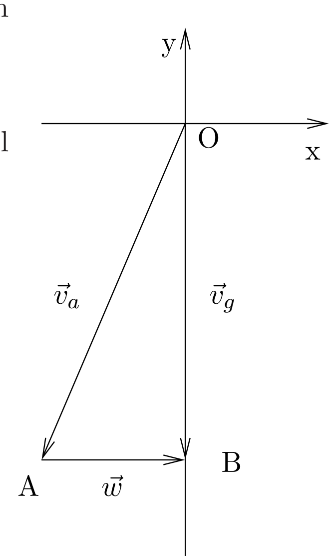
$$|\vec{v}_g| = |\overrightarrow{OB}| - 4 = \sqrt{4177} - 4 \approx 60.63\text{mph}.$$



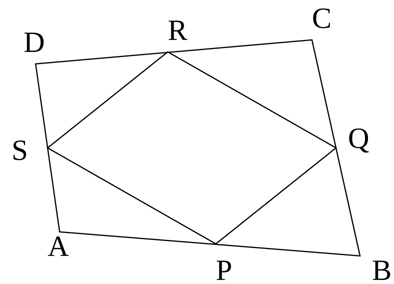
At 6500 ft, the computation is even simpler, as shown in the diagram to the right. Now $\vec{w} = (25, 0)$ and $\triangle OAB$ is a right triangle. So

$$|\vec{v}_g| = \sqrt{|\vec{v}_a|^2 - |\vec{w}|^2} = \sqrt{65^2 - 25^2} = 60\text{mph}.$$

Hence Smaug is better off flying at 4500 ft as his ground speed will be a little higher.



5. (10 pts) **Extra credit problem.** Let A, B, C, D be four points in the plane that form a quadrilateral (i.e. no three of them lie on the same line). Let P, Q, R, and S be the midpoints of the four sides of this quadrilateral. Use vectors to prove that P, Q, R, and S are the vertices of a parallelogram.



Note that

$$\overrightarrow{PS} = \overrightarrow{AS} - \overrightarrow{AP} = \frac{\overrightarrow{AD}}{2} - \frac{\overrightarrow{AB}}{2} = \frac{\overrightarrow{AD}}{2} + \frac{\overrightarrow{BA}}{2} = \frac{\overrightarrow{BA} + \overrightarrow{AD}}{2} = \frac{\overrightarrow{BD}}{2}$$

and

$$\overrightarrow{QR} = \overrightarrow{CD} - \overrightarrow{CB} = \frac{\overrightarrow{CD}}{2} - \frac{\overrightarrow{CB}}{2} = \frac{\overrightarrow{CD}}{2} + \frac{\overrightarrow{BC}}{2} = \frac{\overrightarrow{BC} + \overrightarrow{CD}}{2} = \frac{\overrightarrow{BD}}{2}.$$

Hence $\vec{PS} = \vec{QR}$, so PS and QR are parallel. A similar argument shows $\vec{PQ} = \vec{SR}$, so PQ and SR are also parallel. Therefore $PQRS$ is a parallelogram.