Math 244 Exam 2 Solutions

Apr 1, 2015

1. (10 pts) Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Is the following statement true or false? Justify your answer.
   If $\vec{v}$ is a non-zero vector and $\vec{v} \times \vec{u} = \vec{v} \times \vec{w}$, then $\vec{u} = \vec{w}$.

   The statement is false. To construct a counterexample, consider

$$\vec{v} \times \vec{u} = \vec{v} \times \vec{w} \iff 0 = \vec{v} \times \vec{u} - \vec{v} \times \vec{w} = \vec{v} \times (\vec{u} - \vec{w}).$$

Since $\vec{v} \neq \vec{0}$, we have $\vec{v} \times (\vec{u} - \vec{w}) = 0$ when $\vec{u} - \vec{w} = \vec{0}$ or when $\vec{u} - \vec{w} \parallel \vec{v}$. So choose any two vectors $\vec{u} \neq \vec{w}$, say $\vec{u} = (1, 0, 0)$ and $\vec{w} = (0, 1, 0)$ and set $\vec{v} = \vec{u} - \vec{w} = (1, -1, 0)$. Then $\vec{v}$ is certainly parallel to $\vec{u} - \vec{w}$. Sure enough

$$\begin{aligned} \vec{v} \times \vec{u} &= (1, -1, 0) \times (1, 0, 0) = (0, 0, 1) \\ \vec{v} \times \vec{w} &= (1, -1, 0) \times (0, 1, 0) = (0, 0, 1) \end{aligned}$$

2. (10 pts) Let f and g be function on 3-space (i.e. $f, g : \mathbb{R}^3 \to \mathbb{R}$). Suppose f is differentiable and that

$$\operatorname{grad} f(x, y, z) = (x\vec{i} + y\vec{j} + z\vec{k})g(x, y, z).$$

Explain why f must be constant on any sphere centered at the origin.

Notice that if you are at the point (x, y, z), then the vector $x\vec{i} + y\vec{j} + z\vec{k}$ points from your position directly away from the origin. So the vector $(x\vec{i} + y\vec{j} + z\vec{k})g(x, y, z)$ points either away from the origin or toward the origin depending on whether $g(x, y, z)$ is positive or negative. Or it could be $\vec{0}$ if $g(x, y, z) = 0$. So either $\vec{\nabla} f = \vec{0}$ or it is perpendicular to the sphere centered at the origin whose surface contains (x, y, z). Thus if $\vec{u}$ is a unit vector that is tangent to the surface of the sphere, $f_{\vec{u}} = \vec{\nabla} f \cdot \vec{u} = 0$. This means the rate of change of f is 0 in any direction along on the surface of the sphere. So f must be constant on such a sphere.

3. (10 pts) Let

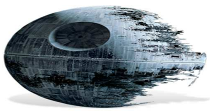
$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

and $\vec{u} = (u_1, u_2)$ a unit vector. Use the definition of the directional derivative to find $f_{\vec{u}}(0, 0)$. Note that you cannot use the shortcut $f_{\vec{u}} = \vec{\nabla} f \cdot \vec{u}$ because this function turns out not to be differentiable at $(0, 0)$.

$$\begin{aligned} f_{\vec{u}}(0, 0) &= \lim_{h \to 0} \frac{f(0 + hu_1, 0 + hu_2) - f(0, 0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{(hu_1)^2 (hu_2)}{(hu_1)^2 + (hu_2)^2} - 0}{h} \\ &= \lim_{h \to 0} \frac{\frac{h^3 u_1^2 u_2}{h^2 (u_1^2 + u_2^2)}}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{u_1^2 u_2}{u_1^2+u_2^2} && \text{by canceling } h \\
&= \frac{u_1^2 u_2}{u_1^2+u_2^2} \\
&= u_1^2 u_2 && \text{since } u_1^2+u_2^2=1
\end{aligned}
$$

4. (10 pts) After the destruction of its first two Death Stars, the Galactic Empire decides to build an improved version, whose shape is elliptical and is given by the equation $x^2+2y^2+4z^2=40000$. The plans call for a flat landing platform for imperial destroyers at the point $(100,100,50)$ that is tangent to the surface of the Death Star. Help the imperial engineers with their evil plan by finding the equation of the tangent plane to the surface at the point given. (Hint: if all else fails, use the Force. I mean the dark side of it.)



Treat this elliptical surface as the level surface $f(x,y,z)=40000$ of the function $f(x,y,z)=x^2+2y^2+4z^2$. Since $\vec{\nabla} f$ is perpendicular to the level surface,

$$\vec{\nabla} f(100,100,50) = (2x,4y,8z)|_{(100,100,50)} = (200,400,400)$$

is a normal vector to the tangent plane to the level surface. Hence

$$200(x-100)+400(y-100)+400(z-50)=0$$

is an equation of the tangent plane.

5. (10 pts) **Extra credit problem.** In Calculus I, you learned–and hopefully proved–that if the derivative of $f(x)$ exists at x_0 then f must be continuous at x_0. By analogy, you might think that if $f(x,y)$ is a function whose partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at a point (x_0,y_0) then f must be continuous at (x_0,y_0). This is not true. Find a counterexample and prove that it is a counterexample.

The key to constructing a counterexample is to notice that the partial derivatives only say something about the values of the function along lines parallel to the coordinate axes. Outside those directions, the function can do anything. So for example, a function which is 0 on the x-axis and y-axis would have $f_x(0,0)=0$ and $f_y(0,0)=0$. But we can easily make it discontinuous at $(0,0)$ by letting $f(x,y)=1$ everywhere else. So

$$f(x,y)=\begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{if } x\neq 0 \text{ and } y\neq 0 \end{cases}$$

is a function whose partial derivatives exist at 0:

$$
\begin{aligned}
f_x(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{0-0}{x} = 0 \\
f_y(0,0) &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0}\frac{0-0}{y} = 0
\end{aligned}
$$

But f is clearly not continuous at $(0,0)$ as

$$
\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) &= \lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0 \\
\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) &= \lim_{x\to 0} f(x,x) = \lim_{x\to 0} 1 = 1
\end{aligned}
$$

disagree.