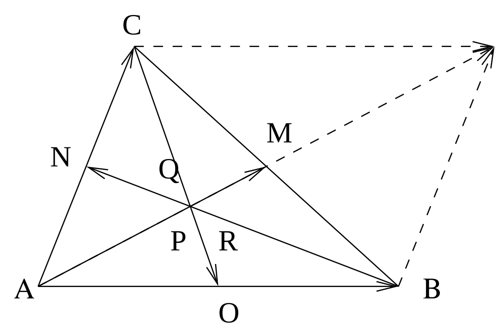
Math 244 Final Exam Solutions

May 8, 2015

1. (10 pts) Use vectors to show that the medians of a triangle intersect at a point 1/3 of the way along each median from the side it bisects. (Hint: For each median, use the position vectors of the vertices $\vec{A}$, $\vec{B}$, and $\vec{C}$ to express the position vector of the point 1/3 of the way from the side the median bisects. Notice that the three expressions you get this way are the same.)

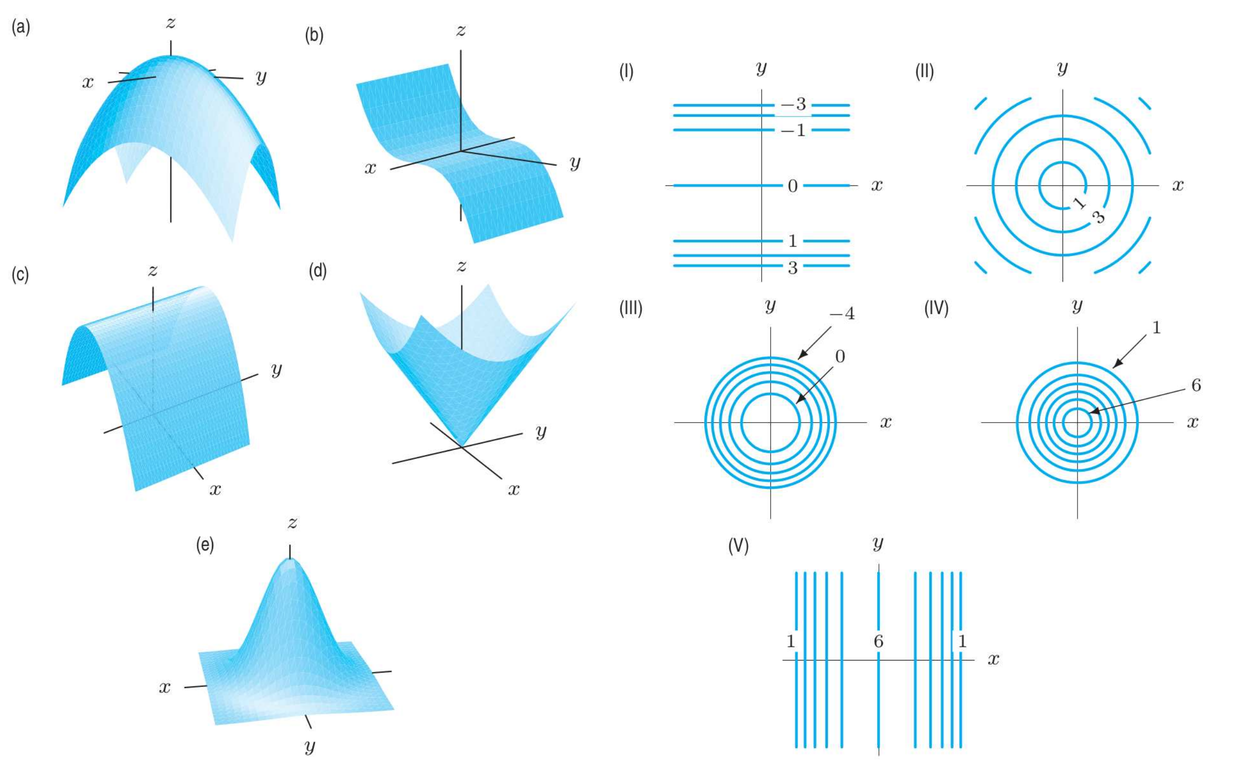
   Let A, B, C be the vertices of the triangle and $\vec{A}, \vec{B}, \vec{C}$ position vectors to those. Let M, N, O be the midpoints of the sides, and P, Q, R the points on the medians 2/3 of the way from the vertex to the corresponding midpoint.

$$\begin{aligned}
\overrightarrow{AB} &= \vec{B} - \vec{A} \\
\overrightarrow{AC} &= \vec{C} - \vec{A} \\
\overrightarrow{AM} &= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) = \frac{1}{2}(\vec{B} + \vec{C} - 2\vec{A}) \\
\overrightarrow{AP} &= \frac{2}{3}\overrightarrow{AM} = \frac{1}{3}(\vec{B} + \vec{C} - 2\vec{A}) \\
\vec{P} &= \vec{A} + \overrightarrow{AP} = \frac{1}{3}(\vec{B} + \vec{C} + \vec{A})
\end{aligned}$$



   By analogous computations, $\vec{Q} = \frac{1}{3}(\vec{B}+\vec{C}+\vec{A})$ and $\vec{R} = \frac{1}{3}(\vec{B}+\vec{C}+\vec{A})$. Since $\vec{P} = \vec{Q} = \vec{R}$, we can conclude $P = Q = R$.

2. (10 pts) Match the surfaces (a)-(e) with the contour diagrams in (I)-(V) below. Be sure to justify your answer.



Notice that the function in (b) depends only on y and not on x, so its contours must be lines parallel to the x-axis. The only such diagram is (I). Similarly, the function in (c) does not depend on y, so its contours are lines parallel to the y-axis, i.e. diagram (V). The graphs in (a), (d), and (e) have rotational symmetry about the z-axis, hence their contour diagrams are concentric circles. The function in (a) is the only one of the three whose values are negative and decrease with distance from the origin. So a matches (III). The function in (d) has value 0 at the origin and then the levels increase with distance from the origin. (II) is the only such diagram. Finally, the function in (e) has a positive value at the origin and its values decrease to 0 with distance from the origin, which is consistent with the only remaining diagram in (IV). To summarize, the matches are

$$a \leftrightarrow III, b \leftrightarrow I, c \leftrightarrow V, d \leftrightarrow II, e \leftrightarrow IV.$$

3. (10 pts) Use appropriate coordinates to find the average distance to the origin for points in the ice cream cone region bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$. (Hint: Since it is easy enough to make a mistake in evaluating such an integral, you may want to spend a little time in the end thinking about whether your answer makes intuitive sense.)

Well, I did it again. I copied the problem from the book, and despite reading the exam several times did not notice that the square root sign in $z = \sqrt{x^2 + y^2}$ did not copy. Nobody asked me during the exam if there was a typo. Most of you corrected the error, I guess either by recognizing the typo or–like me–by seeing what you wanted to see. Of course, I accepted either interpretation, and gave you credit for solving the problem you chose to solve. If you did work with the paraboloid $z = x^2 + y^2$ instead of the cone $z = \sqrt{x^2 + y^2}$, you got more complicated–but technically still doable with Calc II skills–integrals than intended. I took that into account when grading your solution and gave you generous credit for making progress in evaluating that integral, even if you did not reach a complete and correct answer. That said, here is the solution of the original problem from the textbook with the cone $z = \sqrt{x^2 + y^2}$.

The natural coordinates to use are spherical coordinates. In these, the upper boundary of the region is the sphere $\rho = \sqrt{8}$. The cone, which bounds the remaining sides has sides that form 45° angles with the positive z-axis. We know this because its cross-section in the xz-plane is $z = \sqrt{x^2} = |x|$ and the slope of the $|x|$ function is 1 for $x > 0$. So in spherical coordinates, the region is

$$R = \left\{(\rho, \phi, \theta) \mid 0 \leq \rho \leq \sqrt{8}, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \theta < 2\pi\right\}.$$

Now, to find the average of the distance from the origin, we need to integrate the distance ρ over this region and divide by the volume of the region.

$$\begin{aligned}
\int_R \rho\, dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho\rho^2 \sin(\phi)\, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^4}{4} \sin(\phi)\bigg|_{\rho=0}^{\rho=\sqrt{8}} d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{64 - 0}{4} \sin(\phi)\, d\phi d\theta \\
&= \int_0^{2\pi} -16\cos(\phi)|_0^{\pi/4}\, d\theta
\end{aligned}$$

$$= -32\pi\left(\frac{1}{\sqrt{2}} - 1\right) = 16\pi(2 - \sqrt{2})$$

$$\int_R 1\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2\sin(\phi)\,d\rho d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin(\phi)\Big|_{\rho=0}^{\rho=\sqrt{8}} d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{8\sqrt{8}-0}{3}\sin(\phi)\,d\phi d\theta$$

$$= \int_0^{2\pi} -\frac{16\sqrt{2}}{3}\cos(\phi)\Big|_0^{\pi/4} d\theta$$

$$= -2\pi\frac{16\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}} - 1\right) = \frac{32}{3}\pi(\sqrt{2}-1)$$

And the average distance is

$$\frac{16\pi(2-\sqrt{2})}{\frac{32}{3}\pi(\sqrt{2}-1)} = \frac{3}{\sqrt{2}} \approx 2.12.$$

Is that a reasonable number? The radius of the sphere is $\sqrt{8} \approx 2.83$, so the averag should be between 0 and 2.83. Much of the volume of the region is at the outer edge, so it is reasonable to get a number that is much closer to 2.83 than to 0.

In case you are curious what the integrals look like if the bottom boundary is in fact the paraboloid $z = x^2 + y^2$, here they are. First, in spherical coordinates:

$$\int_R \rho\,dV = \int_0^{2\pi}\int_0^{\alpha}\int_0^{\sqrt{8}} \rho^3\sin(\phi)\,d\rho d\phi d\theta + \int_0^{2\pi}\int_{\alpha}^{\pi/2}\int_0^{\cot(\phi)\csc(\phi)} \rho^3\sin(\phi)\,d\rho d\phi d\theta$$

$$\int_R 1\,dV = \int_0^{2\pi}\int_0^{\alpha}\int_0^{\sqrt{8}} \rho^2\sin(\phi)\,d\rho d\phi d\theta + \int_0^{2\pi}\int_{\alpha}^{\pi/2}\int_0^{\cot(\phi)\csc(\phi)} \rho^2\sin(\phi)\,d\rho d\phi d\theta$$

where $\alpha = \arcsin\left(\frac{\sqrt{\sqrt{33}-1}}{4}\right)$. You can do these integrals by using the identity $\cot^2(\phi) + 1 = \csc^2(\phi)$ and substution. In cylindrical coordinates:

$$\int_R \sqrt{r^2+z^2}\,dV = \int_0^{2\pi}\int_0^{a}\int_{r^2}^{\sqrt{8-r^2}} r\sqrt{r^2+z^2}\,dz dr d\theta$$

$$\int_R 1\,dV = \int_0^{2\pi}\int_0^{a}\int_{r^2}^{\sqrt{8-r^2}} r\,dz dr d\theta$$

where $a = \sqrt{\frac{\sqrt{33}-1}{2}}$. This may look simpler at the first, but I think the integrals are trickier to do. You will need some trig substitutions and probably a few rounds of integration by parts. The result of the first integral is about 20.66, the volume is about 10.59, and the average is about 1.95.

4. (a) (3 pts) Let $f(x_1, x_2, \ldots, x_n)$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$. State the definition of the directional derivative $f_{\vec{u}}(x_1, \ldots, x_n)$.

If $\vec{u} = (u_1, \ldots, u_n)$, then

$$f_{\vec{u}}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1 + hu_1, \ldots, x_n + hu_n) - f(x_1, \ldots, x_n)}{h}.$$

(b) (3 pts) Let $f(x, y)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$. Define what it means for f to be differentiable at a point (x_0, y_0).

f is differentiable at (x_0, y_0) if there is a linear function $L(x, y)$ such that $f(x, y) \approx L(x, y)$ is a good approximation in the sense that the error $E(x, y) = f(x, y) - L(x, y)$ satisfies

$$\lim_{(x,y) \to (x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

(c) (10 pts) Now, let $f(x, y)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$ and $\vec{u}$ a unit vector in $\mathbb{R}^2$. Prove that if f is differentiable at (x_0, y_0) then

$$f_{\vec{u}}(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u}.$$

Since f is differentiable at (x_0, y_0), we know by Example 1 in 14.8 that it has a good linear approximation $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ such that $E(x, y) = f(x, y) - L(x, y)$ satisfies

$$\lim_{(x,y) \to (x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

By the definition of the directional derivative,

$$f_{\vec{u}}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}.$$

Substitute

$$\begin{aligned} f(x_0 + hu_1, y_0 + hu_2) &= L(x_0 + hu_1, y_0 + hu_2) + E(x_0 + hu_1, y_0 + hu_2) \\ &= f(x_0, y_0) + f_x(x_0, y_0)(x_0 + hu_1 - x_0) + \\ &\quad f_y(x_0, y_0)(y_0 + hu_2 - y_0) + E(x_0 + hu_1, y_0 + hu_2) \\ &= f(x_0, y_0) + f_x(x_0, y_0)hu_1 + f_y(x_0, y_0)hu_2 + E(x_0 + hu_1, y_0 + hu_2) \end{aligned}$$

to get

$$\begin{aligned} &f_{\vec{u}}(x_0, y_0) = \\ &= \lim_{h \to 0} \frac{f(x_0, y_0) + f_x(x_0, y_0)hu_1 + f_y(x_0, y_0)hu_2 + E(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h} \\ &= \lim_{h \to 0} \frac{f_x(x_0, y_0)hu_1 + f_y(x_0, y_0)hu_2 + E(x_0 + hu_1, y_0 + hu_2)}{h} \\ &= \lim_{h \to 0} \left[f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 + \frac{E(x_0 + hu_1, y_0 + hu_2)}{h} \right] \\ &= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 + \lim_{h \to 0} \frac{E(x_0 + hu_1, y_0 + hu_2)}{h} \end{aligned}$$

provided that the last limit on the right-hand side exists. We will show it does and is equal to 0. Notice that once that is done, we have the result we set out to prove

$$f_{\vec{u}}(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 = \vec{\nabla} f(x_0, y_0) \cdot \vec{u}.$$

As $h \to 0$, we have $(x_0 + hu_1, y_0 + hu_2) \to (x_0, y_0)$. So

$$0 = \lim_{h \to 0} \frac{E(x_0 + hu_1, y_0 + hu_2)}{\sqrt{(x_0 + hu_1 - x_0)^2 + (y_0 + hu_2 - y_0)^2}}.$$

Since $u_1^2 + u_2^2 = 1$

$$\sqrt{(hu_1)^2 + (hu_2)^2} = \sqrt{h^2(u_1^2 + u_2^2)} = \sqrt{h^2} = |h|.$$

Hence

$$0 = \lim_{h \to 0} \frac{E(x_0 + hu_1, y_0 + hu_2)}{|h|}.$$

This is not quite what we wanted because of the absolute value. Let's investigate the one-sided limits as $h \to 0^+$ and $h \to 0^-$. These are of course also 0 since they are special cases of the two-sided limit.

$$0 = \lim_{h \to 0^+} \frac{E(x_0 + hu_1, y_0 + hu_2)}{|h|} = \lim_{h \to 0^+} \frac{E(x_0 + hu_1, y_0 + hu_2)}{h}$$

where the absolute value sign is not needed since $h > 0$. Similarly,

$$0 = \lim_{h \to 0^-} \frac{E(x_0 + hu_1, y_0 + hu_2)}{|h|} = \lim_{h \to 0^-} \frac{E(x_0 + hu_1, y_0 + hu_2)}{-h}$$

$$\implies \lim_{h \to 0^-} \frac{E(x_0 + hu_1, y_0 + hu_2)}{h} = 0.$$

We have just shown that

$$\lim_{h \to 0^+} \frac{E(x_0 + hu_1, y_0 + hu_2)}{h} = \lim_{h \to 0^-} \frac{E(x_0 + hu_1, y_0 + hu_2)}{h} = 0.$$

Therefore the two-sided limit

$$\lim_{h \to 0} \frac{E(x_0 + hu_1, y_0 + hu_2)}{h}$$

must also converge to 0.

5. In the 24th century, humans realize the dream of interstellar travel by inventing quantum teleportation, which allows them to move people and goods large distances at speeds well above the speed of light. However, a few challenges remain. Quantum teleportation can only move objects along a straight line. If two objects that are being teleported get too close to each other, there is a danger that the particles that comprise one could tunnel over the short distance that separates them and mix with the particles of the other. This is highly undesirable. If you have seen The Fly, you know why. For two teleportation paths to be a safe distance from each other, they must be at least 6 km apart. Transportation engineers are proposing one teleportation corridor along the line whose parametric equation is

$$\vec{r}_1(t) = (x_1, y_1, z_1) = (2t - 1, t + 4, 2t - 5)$$

where $t \in \mathbb{R}$ and the coordinates are in km; and another one whose parametric equation is

$$\vec{r}_2(s) = (x_2, y_2, z_2) = (2s + 4, -2s - 2, s + 8)$$

where $s \in \mathbb{R}$. They would like to compute how far apart these two corridors are. Your job is to help them.

(a) (3 pts) First, find the Euclidean distance between the point on the first line at t and the point on the second line at s. Notice that this distance is function of the two variables t and s. Simplify the expression.

The distance is

$$\begin{aligned}
d &= |\vec{r}_1(t) - \vec{r}_2(s)| \\
&= \sqrt{(2t-1-(2s+4))^2 + (t+4-(-2s-2))^2 + (2t-5-(s+8))^2} \\
&= \sqrt{(2t-2s-5)^2 + (t+2s+6)^2 + (2t-s-13)^2} \\
&= \sqrt{9s^2 + 9t^2 - 8st + 70s - 60t + 230}
\end{aligned}$$

(b) (7 pts) You will now find the mimimum of the function from part (a). Instead of minimizing the distance, it is easier to minimize the square of the distance. Use your knowledge of multivariable calculus to find the global minimum of the square of the distance function from part (a).

Since the square root function is strictly increasing, finding the smallest nonnegative number to put under the square root will minimize the value of the square root. So instead of minimizing the distance, we can minimize the square of the distance. Let

$$f(s,t) = 9s^2 + 9t^2 - 8st + 70s - 60t + 230.$$

Then

$$\vec{\nabla} f = (18s - 8t + 70, 18t - 8s - 60).$$

Since this exists everywhere, the only kind of critical point is where $\vec{\nabla} f = 0$. Let us find those:

$$\begin{aligned}
18s - 8t + 70 = 0 &\implies 4(18s - 8t + 70) = 0 \implies 72s - 32t + 280 = 0 \\
18t - 8s - 60 = 0 &\implies 9(18t - 8s - 60) = 0 \implies 162t - 72s - 540 = 0
\end{aligned}$$

Hence

$$\begin{aligned}
&0 = 72s - 32t + 280 + 162t - 72s - 540 = 130t - 260 \implies t = 2 \\
&0 = 18t - 8s - 60 \implies 8s = 18t - 60 = 36 - 60 = -24 \implies s = -3
\end{aligned}$$

This is the only critical point. It must be the global min for reasons we will spell out in part (c). At $(-3,2)$

$$d = \sqrt{f(-3,2)} = \sqrt{65} \approx 8.06 \text{ km}.$$

(c) (3 pts) How do you know that value you found is a global mimimum?

First, if f has a global minimum then it must also be a local minimum. Now f is a polynomial function, hence differentiable everywhere. So at any local minimum, its gradient must be 0, otherwise the gradient would point in the direction of fastest increase and moving in the opposite direction, the values of the function would have to decrease, which is clearly impossible at a local min. So if there is a global minimum, it must be the point we found in (b). So how do we know there must be a global minimum? Think about the original problem. Since $f(s,t)$ is the square of the distance of two points that are sliding along two lines, there must be a mimimum value somewhere. Otherwise the two lines would have to inch closer and closer to each other somewhere without ever reaching a minimum distance. Lines don't do that.

(d) (1 pt) Are the two teleportation corridors a safe distance from each other?

Since the two lines are more than 6 km apart, they are a distance from each other.

6. **Extra credit problem.** Let $f(x,y) = |xy|$.

(a) (6 pts) At what points (x, y) do the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ exist?

Notice that at any point (x, y) that is not on the coordinate axes, has a small enough neighborhood that it does not contain any points on the axes, so $f(x, y) = xy$ or $f(x, y) = -xy$ within that entire neighborhood. Therefore its partial derivatives at such a point (x, y) are the same as the partial derivatives of $f(x, y) = xy$ or $f(x, y) = -xy$. That is if $x, y > 0$ or $x, y < 0$ then $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$. If $x > 0$ and $y < 0$ or vice versa then $\frac{\partial f}{\partial x} = -y$ and $\frac{\partial f}{\partial y} = -x$.
Now,

$$f_x(x, 0) = \lim_{h \to 0} \frac{f(x + h, 0) - f(x, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Similarly, $f_y(0, y) = 0$. If $x \neq 0$

$$f_y(x, 0) = \lim_{h \to 0} \frac{f(x, h) - f(x, 0)}{h} = \lim_{h \to 0} \frac{|xh| - 0}{h} = |x| \lim_{h \to 0} \frac{|h|}{h}$$

does not exist as $|h|/h = \pm 1$ depending on whether $h \to 0^+$ or $h \to o^-$. That is the left and the right limits disagree. Analogously, $f_x(0, y)$ does not exist when $y \neq 0$.
To sum it up, the partial derivatives whenever $xy \neq 0$ and at $(0, 0)$, but $\frac{\partial f}{\partial x}$ does not exist elsewhere on the y-axis and $\frac{\partial f}{\partial y}$ does not exist elsewhere on the x-axis.

(b) (9 pts) At what points (x, y) is f differentiable?

First, notice that wherever f is differentiable, its partial derivatives must exist. Therefore f is certainly not differentiable at any point on the axes, except possibly at $(0, 0)$. If (x, y) is not on one of the axes, then there is always a small enough neighborhood of (x, y) that it does not include any point on the axes. Within this neighborhood $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist and are continuous, since they are equal to y and x or $-y$ and $-x$ respectively. Therefore f is differentiable at these points by Theorem 14.2.
As for the origin, if f is differentiable there then its local linearization must be

$$L(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 0 + 0x + 0y = 0$$

by Example 1 in 14.8. The error function is $E(x, y) = f(x, y) - L(x, y) = |xy|$. Consider

$$\lim_{(h,k) \to (0,0)} \frac{E(0 + h, 0 + k)}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \to (0,0)} \frac{|hk|}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \to (0,0)} \frac{|h||k|}{\sqrt{h^2 + k^2}}.$$

We want to see if this limit is 0. This is a $\delta - \epsilon$ argument that is not hard to do. Let $\epsilon > 0$. Choose $\delta = \epsilon$. If (h, k) is closer to $(0, 0)$ than δ, then $\sqrt{h^2 + k^2} < \delta$. Now notice

$$h^2 + k^2 \geq h^2 \implies \sqrt{h^2 + k^2} \geq \sqrt{h^2} = |h| \implies 0 \leq \frac{|h|}{\sqrt{h^2 + k^2}} \leq 1$$

Hence

$$0 \leq \frac{|h||k|}{\sqrt{h^2 + k^2}} \leq |k| \leq \sqrt{h^2 + k^2} < \delta = \epsilon.$$

This shows that

$$\lim_{(h,k) \to (0,0)} \frac{E(h, k)}{\sqrt{h^2 + k^2}} = 0$$

and therefore f is differentiable at $(0, 0)$.